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NEW ITERATIVE METHODS OF KRASNOSEL'SKII-MANN TYPE FOR THE MULTIPLE-SETS SPLIT FEASIBILITY PROBLEM

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Abstract

In this paper, for solving the multiple-sets split feasibility problem (MSSFP) in Hilbert spaces, a new iterative method of Krasnosel'skii-Mann type and its combination with the steepest-descent algorithm are presented. In particular, the step size in these methods is calculated directly from the iteration procedure without prior knowledge of operator norms. A numerical example is given for illustrating the introduced methods.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces with inner products and norms, denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, and let A be a bounded linear mapping from H_1 into H_2. Let C_i and Q_j be closed convex subsets in H_1 and H_2, respectively, for each $i \in J_1 = \{1, 2, \cdots, p\}$ and $j \in J_2 = \{1, 2, \cdots, r\}$ where p and r are two fixed positive integers.

The MSSFP is to find a point

$$x \in C := \cap_{i \in J_1} C_i \quad \text{such that} \quad Ax \in Q := \cap_{j \in J_2} Q_j. \tag{1.1}$$

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Denote by Γ the solution of (1.1). Throughout this paper, we assume that $\Gamma \neq \emptyset$.

Problem (1.1) was first introduced by Censor and Elfving [4] in finite-dimensional Hilbert spaces for modeling inverse problems that arise from phase retrievals and in image reconstruction [2]. Recently, the MSSFP can also be used to model the intensity-modulated radiation therapy [5-8]. Many iterative methods have been developed to solve this problem. See, for example, [1, 3, 9-11, 13-16, 20-22, 25-28, 30-35, 37, 40-44] and references therein.

In [5], Censor *et al* defined the proximity function $p(x)$ to measure the distance of a point $x \in H_1$ to all sets C_i and Q_j by

$$p(x) = g(x)+q(x),\ g(x) = \frac{1}{2}\sum_{i=1}^{p} \alpha_i \|(I-P_{C_i})x\|^2,\ q(x) = \frac{1}{2}\sum_{j=1}^{r} \beta_j \|(I-P_{Q_j})Ax\|^2,$$

where I denotes the identity mapping in H_m, $m = 1, 2$, $\alpha_i > 0$, $\beta_j > 0$ for all i and j, respectively, with $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{r} \beta_j = 1$, and considered the constrained MSSFP as follows:

$$\text{find } z_* \in \Omega \quad \text{such that } z_* \quad \text{solves} \ \ (1.1), \tag{1.2}$$

where Ω is an auxiliary, simple, nonempty, closed and convex subset in H_1 such that $\Gamma \cap \Omega \neq \emptyset$. For solving (1.2), they proposed the projection-gradient method,

$$x^{k+1} = P_\Omega(x^k - s\triangledown p(x^k)),\ x^1 \in H_1, \tag{1.3}$$

for all $k \in \mathbb{N}_+$, the set of all positive integers, where s is a positive number and

$$\triangledown p(x) = \sum_{i=1}^{p} \alpha_i (I - P_{C_i})x + \sum_{j=1}^{r} \beta_j A^*(I - P_{Q_j} A)x, \tag{1.4}$$

where A^* denotes the adjoint of A. They proved global convergence of (1.3)-(1.4) under condition $0 < s < 2/L$ with $L = \sum_{i=1}^{p} \alpha_i + \|A\|^2 \sum_{j=1}^{r} \beta_j$, being the Lipschitz constant of $\triangledown p(x)$. In infinite dimensional Hilbert spaces, basing on the Krasnosel'skii-Mann algorithm (see, [24]),

$$x^{k+1} = (1 - \gamma_k)x^k + \gamma_k T x^k,$$

to find a fixed point of a self-nonexpansive mapping T of a closed convex subset, Xu [37] introduced some weak convergent methods, one of which is a method of Krasnosel'skii-Mann type,

$$\begin{aligned} x^{k+1} = (1 - \gamma_k)x^k + \gamma_k P_{\Omega_k}\Bigg(x^k - s\Bigg(&\sum_{i=1}^{p} \alpha_i (I - P_{C_{ik}})x^k \\ &+ \sum_{j=1}^{r} \beta_j A^*(I - P_{Q_{jk}})Ax^k\Bigg)\Bigg), \end{aligned} \tag{1.5}$$

where $\gamma_k \in (0,1)$ with $\sum_{k=1}^{\infty} \gamma_k(1-\gamma_k) = \infty$ and the perturbations Ω_k, C_{ik}, Q_{jk} of the sets Ω, C_i, Q_j satisfy some approximation properties. Next, He et al [20], combining three iterative methods in [37] with the Krasnosel'skii-Mann algorithm, obtained the following weakly convergent iterative methods:

$$\begin{aligned} x^{k+1} &= (1-\gamma_k)x^k + \gamma_k T_p T_{p-1} \cdots T_1 x^k, T_i = P_{C_i}(I - s\nabla q), \\ x^{k+1} &= (1-\gamma_k)x^k + \gamma_k \sum_{i=1}^{p} \alpha_i P_{C_i}\left(I - s\sum_{j=1}^{r} \beta_j A^*(I - P_{Q_j})A\right)x^k, \\ x^{k+1} &= (1-\gamma_k)x^k + \gamma_k T_{[k+1]} x^k, \end{aligned} \tag{1.6}$$

where $T_{[k]} = T_{kmod\ p}$, the mod function takes values in $\{1, 2, \cdots, p\}$, α_i satisfies condition

(α) $\alpha_i > 0$ for each $i \in J_1$ such that $\sum_{i=1}^{p} \alpha_i = 1$,

$0 < s < 2/L$ with $L = \|A\|^2 \sum_{j=1}^{r} \beta_j$. A strong convergent method of Krasnosel'skii-Mann type,

$$x^{k+1} = (1-\gamma_k)x^k + \gamma_k P_{C_{[k]}}[(1-t_k)(I - \tau_k A^*(I - P_{Q_{[k]}})A)]x^k \tag{1.7}$$

in the case that $p = r$, was introduced by Wang et al [33], where t_k has the properties

(t) $t_k \in (0,1)$ for all $k \in \mathbb{N}_+$, $\lim_{k\to\infty} t_k = 0$ and $\sum_{k=1}^{\infty} t_k = \infty$,

$P_{C_{[k]}} = P_{C_{kmod\ p}}$ and $P_{Q_{[k]}} = P_{Q_{kmod\ p}}$ with $0 < \liminf_{k\to\infty} \tau_k \le \limsup_{k\to\infty} \tau_k < 1/\|A\|^2$. In order to obtain a strong convergent iterative method, Dang and Gao [13] combined the Krasnosel'skii-Mann algorithm with the Byrne's CQ algorithm [2] for the split feasibility problem (SFP), that is (1.1) with $p = r = 1$. Very recently, for solving the SFP Yu et al [41] presented an iterative method of Krasnosel'skii-Mann type,

$$x^{k+1} = (1-\gamma_k)x^k + \gamma_k[P_C - \tau_k A^*(I - P_Q)A]x^k, \tag{1.8}$$

with conditions: $0 < \liminf_{k\to\infty} \gamma_k \le \limsup_{k\to\infty} \gamma_k < 1$ and $\tau_k \in (0, 1/(\gamma_k\|A\|^2))$. A simpler method was presented by Wang [34],

$$x^{k+1} = (1-\gamma_k)x^k + \gamma_k[P_C - A^*(I - P_Q)A]x^k, \tag{1.9}$$

where γ_k has the properties: $\sum_{k=1}^{\infty} \gamma_k = \infty$ and $\sum_{k=1}^{\infty} \gamma_k^2 < \infty$. At this time, Ng. Buong [1] proposed several weak convergent methods

$$x^{k+1} = Tx^k,\ T = P_1(I - sA^*(I - P_2)A), \tag{1.10}$$

where P_1 and P_2 are defined by one of the conditions (c):

(c1) $P_1 = P_{C_p} \cdots P_{C_1}$ and $P_2 = P_{Q_r} \cdots P_{Q_1}$;

(c2) $P_1 = \sum_{i=1}^{p} \alpha_i P_{C_i}$ and $P_2 = \sum_{j=1}^{r} \beta_j P_{Q_j}$;

(c3) $P_1 = P_{C_p} \cdots P_{C_1}$ and $P_2 = \sum_{j=1}^{r} \beta_j P_{Q_j}$;

(c4) $P_1 = \sum_{i=1}^{p} \alpha_i P_{C_i}$ and $P_2 = P_{Q_r} \cdots P_{Q_1}$

with $s \in (0, 1/\|A\|^2)$, (α) and
(β) $\beta_j > 0$ for $1 \leq j \leq r$ such that $\sum_{j=1}^r \beta_j = 1$.

In order to obtain a strong convergence sequence $\{x^k\}$ from (1.10), he also proposed the method

$$x^{k+1} = (I - t_k \mu F)Tx^k, \tag{1.11}$$

where F is an η-strongly monotone and $\tilde{L}$-Lipschitz continuous mapping on H_1, μ is a fixed number in $(0, 2\eta/\tilde{L}^2)$ and t_k has property (t).

Weak and strong convergence theorems for the MSSFP in Banach spaces were obtained in [26, 27] and references therein.

Methods (1.3)-(1.4), (1.5), (1.6), (1.7), (1.8), (1.10) and (1.11) use a fixed step size restricted by the constants which depend on the largest eigenvalue (spectral radius) of the operator A^*A. Computing the largest eigenvalue may be very hard and conservative estimate of the constants usually results in slow convergence. Motivated by a self-adaptive strategy given by He *et al* [19], Zhang *et al* [42], Zhao *et al* [43] and Zhao with Yang [44] proposed several self-adaptive projection-gradient methods. Two modifications of a method in [43] were studied in [10] and [40]. These methods, at each iteration step, need an inner iteration numbers to obtain a suitable step size. To exclude the drawback in solving the SFP, López *et al* [22] suggested a new self-adaptive way to compute directly the step size in each iteration. By considering the constrained optimization problem $\min_{x \in C} q(x)$, they proposed the weakly convergent projection-gradient method

$$x^{k+1} = P_C(x^k - \tau_k \nabla q(x^k)), \tag{1.12}$$

where

$$\tau_k = \frac{\rho_k q(x^k)}{\|\nabla q(x^k)\|^2}, \tag{1.13}$$

(ρ) $\rho_k \in (0, 4)$ for all $k \in \mathbb{N}_+$ and $\liminf_{k\to\infty} \rho_k > 0$.
Here, they also introduced strongly convergent methods, by combining (1.12)-(1.13) with the hybrid method in mathematical programming [29] and Halpern method [18]. Recently, in infinite-dimensional Hilbert spaces, Tang *et al* [30] and Wen *et al* [35] proposed also weakly convergent methods.

It is not difficult to see that when α_i and β_j satisfy conditions (α) and (β), we have

$$\sum_{i=1}^p \alpha_i(I - P_{C_i}) = I - \sum_{i=1}^p \alpha_i P_{C_i} \quad \text{and} \quad \sum_{j=1}^r \beta_j A^*(I - P_{Q_j})A = A^*\left(I - \sum_{j=1}^r \beta_j P_{Q_j}\right)A,$$

since A is a linear mapping. This changes decrease the computational time for methods (1.3)-(1.4), (1.5) and (1.6) because the number of operations for the left-hand side of the equalities at any point is more than that for the right-hand side. By the same reason, it will be better if we can replace $\sum_{i=1}^{p}\alpha_i P_{C_i}$ and $\sum_{j=1}^{r}\beta_j P_{Q_j}$ in the equalities above by $P_{C_p}\cdots P_{C_1}$ and $P_{Q_r}\cdots P_{Q_1}$, respectively, in constructing algorithms for (1.1).

In this paper, motivated by the results in the listed works and the above remarks, we give a new iterative method of Krasnosel'skii-Mann type,

$$x^{k+1} = (1-\gamma_k)x^k + \gamma_k T_k x^k,\ T_k = P_1 - \tau_k A^*(I - P_2)A, \tag{1.14}$$

where P_1 and P_2 are defined by (c1) and (c2), respectively, the parameter γ_k satisfies condition
(γ) $\gamma_k \in [a,b] \subset (0,(p+1)/(2p))$,
τ_k is determined by

$$\tau_k = \frac{\rho_k f(x^k)}{(\|u^k\| + \lambda_k)^2} \tag{1.15}$$

with $u^k = A^*(I-P_2)Ax^k$,
(ρ') $\rho_k \in [\varepsilon,(r+1)/r-\varepsilon]$ for all $k \in \mathbb{N}_+$, where ε is a small positive number,
(λ) $\lambda_k \in [c,d] \subset (0,\infty)$ for all $k \in \mathbb{N}_+$
and $f(x) = \|(I-P_2)Ax\|^2/2$. Next, from (1.14), we can design a strong convergent sequence $\{x^k\}$ by

$$x^{k+1} = (I - t_k\mu F)[(1-\gamma_k)I + \gamma_k T_k]x^k, \tag{1.16}$$

that is a combination of the Krasnosel'skii-Mann type algorithm with the steepest-descent one, where t_k satisfies condition (t).

The rest of this paper is organized as follows. In Section 2, we list some related facts, that will be used in the proof of our results. In Section 3, we prove weak and strong convergences of our methods under suitable conditions. As consequences, we obtain some modifications of the Krasnosel'skii-Mann and Halpern algorithms. Finally, in Section 4, we give some numerical experiments for testing our theoretical results.

2. Preliminaries

Let H be a real Hilbert space with inner product and norm, denoted, respectively, also by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Then,

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle\ \forall x,y \in H.$$

Definitions 2.1 A mapping T from a subset C of H into H is called:
(i) nonexpansive, if $\|Tx - Ty\| \le \|x-y\|$ for all $x,y \in C$;
(ii) contractive, if $\|Tx - Ty\| \le \tilde{a}\|x-y\|$ for a fixed $\tilde{a} \in [0,1)$ and for all $x,y \in C$;

(iii) γ-inverse strongly monotone, if $\gamma\|Tx-Ty\|^2 \leq \langle Tx-Ty, x-y\rangle$ for all $x,y\in C$, where γ is a positive number;
(iv) firmly nonexpansive, if there holds (iii) with $\gamma=1$;
(v) averaged, if $T=(1-\omega)I+\omega N$ for some fixed $\omega\in(0,1)$ and a nonexpansive mapping N and we say T is ω-averaged.

We denote by $Fix(T)$ the set of all fixed points of T, i.e.,

$$Fix(T)=\{x\in C: x=Tx\}.$$

Lemma 2.1 [17] *Let C be a closed convex subset of a real Hilbert space H and let T be a nonexpansive mapping from C into H with $Fix(T)\neq\emptyset$. If $\{x^k\}$ is a sequence in C weakly converging to x and if $(I-T)x^k$ converges strongly to y, then $(I-T)x=y$. In particular, if $y=0$, then $x\in Fix(T)$.*

Lemma 2.2 [38, 12] *We have:*
(i) T is ω-averaged, if and only if $I-T$ is $(1/2\omega)$-inverse strongly monotone;
(ii) Let D be a nonempty set of H and let $m\geq 2$ be an integer. Set

$$\omega = 1\bigg/\left(1+1/\sum_{i=1}^{m}\omega_i/(1-\omega_i)\right),\ \omega_i\in(0,1)\ \forall i\in\{1,2,\cdots,m\}$$

and let $T_i: D\to D$ be ω_i-averaged. Set $T:=T_mT_{m-1}\cdots T_1$. Then, T is ω-averaged;
(iii) Let D,ω_i,T_i be as the above and let α_i be satisfied condition (α) with *$p=m$. Then, the mapping T, defined by $T=\sum_{i=1}^m \alpha_i T_i$, is ω-averaged, where $\omega=\sum_{i=1}^m\omega_i$.*

For a closed convex subset C of H, there exists a mapping $P_C: H$ onto C such that $P_C(x)=\inf_{y\in C}\|y-x\|$ for each $x\in H$. The mapping P_C is called the metric projection onto C. We know that P_C is firmly nonexpansive (hence, nonexpansive), $I-P_C$ is also firmly nonexpansive and

$$\|P_Cx-z\|^2\leq\|x-z\|^2-\|x-P_Cx\|^2\ \forall x\in H, z\in C. \tag{2.1}$$

Recall that a sequence $\{x^k\}$ in H is said to be Fejér monotone with respect to (w.r.t.) a nonempty, closed and convex subset S in H, if

$$\|x^{k+1}-z\|\leq\|x^k-z\|\ \forall k\in\mathbb{N}_+,\ z\in S.$$

Lemma 2.3 [22] *Let S be a nonempty, closed and convex subset in H. If the sequence $\{x^k\}$ is Fejér monotone w.r.t. S, then $\{x^k\}$ converges weakly to a point in S if and only if all weak cluster points of $\{x^k\}$ belong to S.*

A mapping F, defined on H, is said to be η-strongly monotone and $\tilde{L}$-Lipschitz continuous, if F satisfies, respectively, the following conditions:

$$\langle Fx-Fy, x-y\rangle\geq\eta\|x-y\|^2 \quad\text{and}\quad \|Fx-Fy\|\ \leq\tilde{L}\|x-y\|\ \forall x,y\in H,$$

where η and $\tilde{L}$ are fixed positive numbers.

Lemma 2.4 [39] *Let F be an η-strongly monotone and $\tilde{L}$-Lipschitz continuous mapping on a real Hilbert space H. Then, for two fixed numbers $\mu \in (0, 2\eta/\tilde{L}^2)$ and $t \in (0,1)$, we have $\|F^t x - F^t y\| \leq (1 - t\tau)\|x - y\| \quad \forall x, y \in H$, where $F^t = I - t\mu F$ and constant $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\tilde{L}^2)} \in (0,1)$.*

Lemma 2.5 [36] *Let $\{a_k\}$, $\{t_k\}$ and $\{c_k\}$ be sequences of real numbers such that*
(i) $a_{k+1} \leq (1 - t_k)a_k + t_k c_k$;
(ii) $a_k \geq 0$;
(iii) t_k satisfies condition (t);
(iv) $\limsup_{k\to\infty} c_k \leq 0$.
Then, $lim_{k\to\infty} a_k = 0$.

Lemma 2.6 [23] *Let $\{a_k\}$ be a sequence of real numbers with a subsequence $\{k_l\}$ of $\{k\}$ such that $a_{k_l} < a_{k_l+1}$ for all $l \in \mathbb{N}_+$. Then, there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}_+$ such that $m_k \to \infty, a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$ for all (sufficiently large) numbers $k \in \mathbb{N}_+$. In fact, $m_k = \max\{l \leq k : a_l \leq a_{l+1}\}$.*

3. Main Results

We have the following results.
Theorem 3.1 *Let H_1 and H_2 be two real Hilbert spaces and let A be a bounded linear mapping from H_1 into H_2. Let C_i and Q_j be closed convex subsets in H_1 and H_2, respectively, for each $i \in J_1 = \{1, 2, \cdots, p\}$ and $j \in J_2 = \{1, 2, \cdots, r\}$. Assume that $\Gamma \neq \emptyset$ and there hold conditions (γ), (ρ') and (λ). Then, the sequence $\{x^k\}$, defined by* (1.14)-(1.15), *and one of the cases in condition* (c), *as $k \to \infty$, converges weakly to a point in Γ.*

Proof. Clearly, $P_1 z = z$ and $(I - P_2)Az = 0$, for any $z \in \Gamma$.

From (1.14), we deduce immediately that

$$\begin{aligned}\|x^{k+1} - z\|^2 &= \|x^k - z - \gamma_k(x^k - T_k x^k)\|^2 \\ &= \|x^k - z\|^2 - 2\gamma_k\langle x^k - T_k x^k, x^k - z\rangle + \gamma_k^2\|x^k - T_k x^k\|^2 \\ &\leq \|x^k - z\|^2 - 2\gamma_k\langle (I - P_1)x^k, x^k - z\rangle - 2\gamma_k\tau_k\langle u^k, x^k - z\rangle \\ &\quad + 2\gamma_k^2(\|(I - P_1)x^k\|^2 + \tau_k^2\|u^k\|^2).\end{aligned} \tag{3.1}$$

First, we consider the case, when $P_1 = P_{C_p} \cdots P_{C_1}$ and $P_2 = P_{Q_r} \cdots P_{Q_1}$.

Since P_{C_i} is (1/2)-averaged (see, [2]), by Lemma 2.2, P_1 is ω-averaged with $\omega = p/(p+1)$. So, $I - P_1$ is $(1/(2\omega))$-inverse strongly monotone, i.e.,

$$\langle (I - P_1)x^k, x^k - z\rangle \geq \frac{p+1}{2p}\|(I - P_1)x^k\|^2. \tag{3.2}$$

Similarly, we have

$$\begin{aligned}\langle u^k, x^k - z\rangle &= \langle (I-P_2)Ax^k, Ax^k - Az\rangle \\ &\geq \frac{r+1}{2r}\|(I-P_2)Ax^k\|^2 = \frac{r+1}{r}f(x^k).\end{aligned} \tag{3.3}$$

From (3.1), (3.2) and (3.3) it implies that

$$\begin{aligned}\|x^{k+1}-z\|^2 &\leq \|x^k - z\|^2 - \frac{\gamma_k(p+1)}{p}\|(I-P_1)x^k\|^2 - 2\gamma_k\tau_k\frac{r+1}{r}f(x^k) \\ &\quad + 2\gamma_k^2(\|(I-P_1)x^k\|^2 + \tau_k^2\|u^k\|^2) \\ &\leq \|x^k - z\|^2 - 2\gamma_k\left(\frac{p+1}{2p} - \gamma_k\right)\|(I-P_1)x^k\|^2 \\ &\quad - 2\gamma_k\frac{r+1}{r}\frac{\rho_k f^2(x^k)}{(\|u^k\|+\lambda_k)^2} + 2\gamma_k\frac{\rho_k^2 f^2(x^k)}{(\|u^k\|+\lambda_k)^2} \\ &= \|x^k - z\|^2 - 2\gamma_k\left(\frac{p+1}{2p} - \gamma_k\right)\|(I-P_1)x^k\|^2 \\ &\quad - 2\gamma_k\rho_k\left(\frac{r+1}{r} - \rho_k\right)\frac{f^2(x^k)}{(\|u^k\|+\lambda_k)^2},\end{aligned} \tag{3.4}$$

since $b < (p+1)/(2p) \leq 1$ for any integer p. Taking into account of conditions (γ) and (ρ'),

$$\begin{aligned}\|x^{k+1}-z\| &\leq \|x^k - z\|, \\ 2\gamma_k\left(\frac{p+1}{2p} - \gamma_k\right)\|(I-P_1)x^k\|^2 &\leq \|x^k - z\|^2 - \|x^{k+1}-z\|^2, \\ 2\gamma_k\rho_k\left(\frac{r+1}{r} - \rho_k\right)\frac{f^2(x^k)}{(\|u^k\|+\lambda_k)^2} &\leq \|x^k - z\|^2 - \|x^{k+1}-z\|^2.\end{aligned} \tag{3.5}$$

According to the first inequality in (3.5), there exists $\lim_{k\to\infty}\|x^k - z\|$. Therefore, $\{x^k\}$ is bounded, and hence, $\{\|u^k\|\}$ is also bounded. Next, from the second and the last inequalities in (3.5), the existence of $\lim_{k\to\infty}\|x^k - z\|$, the definition of $f(x)$, conditions (ρ') and (λ) with the boundedness of $\{\|u^k\|\}$, it follows that

$$\lim_{k\to\infty}\|(I-P_1)x^k\| = 0 \text{ and } \lim_{k\to\infty} f(x^k) = \frac{1}{2}\lim_{k\to\infty}\|(I-P_2)Ax^k\|/2 = 0. \tag{3.6}$$

Now, we prove that

$$\lim_{k\to\infty}\|(I-P_{C_i})x^k\| = 0 \ \forall i \in J_1 \text{ and } \lim_{k\to\infty}\|(I-P_{Q_j})Ax^k\| = 0 \ \forall j \in J_2. \tag{3.7}$$

Le R be a positive number such that $R \geq \|x^k - z\|$ for all $k \in \mathbb{N}_+$. Using property (2.1) for P_{C_i} with $i = p, p-1, \cdots, 1$, we get that

$$\|P_1 x^k - z\|^2 \leq \|x^k - z\|^2 - \sum_{i=1}^{p} \|S_i x^k - S_{i-1} x^k\|^2,$$

where $S_i = P_{C_i} P_{C_{i-1}} \cdots P_{C_1}$ and $P_{C_0} = I$. On the other hand,

$$\begin{aligned} \|P_1 x^k - z\|^2 &= \|P_1 x^k - x^k\|^2 + \|x^k - z\|^2 + 2\langle P_1 x^k - x^k, x^k - z\rangle \\ &\geq \|P_1 x^k - x^k\|^2 + \|x^k - z\|^2 - 2R\|(I - P_1)x^k\|. \end{aligned} \tag{3.8}$$

From two last inequalities and the first limit in (3.6), we know that

$$\lim_{k\to\infty} \|S_i x - S_{i-1} x^k\| = 0, \; i \in J_1. \tag{3.9}$$

Taking $i = 1$ in (3.9), we have immediately $\lim_{k\to\infty} \|(I - P_{C_1})x^k\| = 0$. In the case that $i = 2$ in (3.9), $\lim_{k\to\infty} \|P_{C_2} P_{C_1} x - P_{C_1} x^k\| = 0$, which together with the conclusion for the case $i = 1$ implies $\lim_{k\to\infty} \|(I - P_{C_2})x^k\| = 0$. Repeating the process to $i = p$, we have the first limit in (3.7). By the second limit in (3.6) and the similar argument as the above, we know that

$$\lim_{k\to\infty} \|\tilde{S}_j A x^k - \tilde{S}_{j-1} A x^k\| = 0,$$

where $\tilde{S}_j = P_{Q_j} P_{Q_{j-1}} \cdots P_{Q_1}$ and $\tilde{S}_0 = I$. Thus, we obtain the second limit in (3.7).

Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_l}\}$ converging weakly to a point $\tilde{z} \in H_1$. From Lemma 2.1, the property of A, and (3.7) we deduce immediately that $\tilde{z} \in \cap_{i=1}^{p} Fix(P_{C_i})$ and $A\tilde{z} \in \cap_{j=1}^{r} Fix(P_{Q_j})$. It means that $\tilde{z} \in \Gamma$. Similarly, we have that every weak cluster point of the set $\{x^k\}$ belongs to Γ. Consequently, from Lemma 2.3 with $S = \Gamma$ and the first inequality in (3.5), it follows that all sequence $\{x^k\}$ converges weakly to a point in Γ.

Now, consider the case when $P_1 = \sum_{i=1}^{p} \alpha_i P_{C_i}$ and $P_2 = \sum_{i=1}^{r} \beta_j P_{Q_j}$ with conditions (α) and (β). Then,

$$\begin{aligned} \langle (I - P_1)x^k, x^k - z\rangle &= \sum_{i=1}^{p} \alpha_i \langle (I - P_{C_i})x^k, x^k - z\rangle \geq \sum_{i=1}^{p} \alpha_i \|(I - P_{C_i})x^k\|^2 \\ &\geq \left\| \sum_{i=1}^{p} \alpha_i (I - P_{C_i})x^k \right\|^2 = \|(I - P_1)x^k\|^2, \\ \langle u^k, x^k - z\rangle &= \langle (I - P_2)Ax^k, Ax^k - Az\rangle \geq \|(I - P_2)Ax^k\|^2. \end{aligned}$$

Thus, instead of (3.4), we have the inequalities

$$\begin{aligned}\|x^{k+1}-z\|^2 &\le \|x^k-z\|^2-2\gamma_k(1-\gamma_k)\|(I-P_1)x^k\|^2\\ &\quad -2\gamma_k\rho_k(2-\rho_k)\frac{f^2(x^k)}{(\|u^k\|+\lambda_k)^2}\\ &\le \|x^k-z\|^2-2\gamma_k(1-\gamma_k)\|(I-P_1)x^k\|^2\\ &\quad -2\gamma_k\rho_k\left(\frac{r+1}{r}-\rho_k\right)\frac{f^2(x^k)}{(\|u^k\|+\lambda_k)^2},\end{aligned}$$

because $(r+1)/r \le 2$ for any $r \in \mathbb{N}_+$. Consequently, $\{x^k\}$ is bounded and we obtain the limits in (3.6). Next, let R be a positive number such that $\|x^k - z\| \le R$. By the convexity of the function $\|x\|^2$ for $x \in H_1$, (2.1) with condition (α),

$$\begin{aligned}\|P_1x^k-z\|^2 &= \left\|\sum_{i=1}^p \alpha_i(P_{C_i}x^k-z)\right\|^2 \le \sum_{i=1}^p \alpha_i\|P_{C_i}x^k-z\|^2\\ &\le \|x^k-z\|^2-\sum_{i=1}^p \alpha_i\|(I-P_{C_i})x^k\|^2\end{aligned}$$

and (3.8), we know that

$$\frac{1}{R}\sum_{i=1}^p \alpha_i\|(I-P_{C_i})x^k\|^2 \le \|(I-P_1)x^k\|.$$

Therefore, we get the first limit in (3.7). By the similar argument, we also get the second limit in (3.7). The cases, when $P_1 = P_{C_p}\cdots P_{C_1}$ with $P_2 = \sum_{i=1}^r \beta_j P_{Q_j}$ and $P_1 = \sum_{i=1}^p \alpha_i P_{C_i}$ with $P_2 = P_{Q_r}\cdots P_{Q_1}$, are similar. The proof is completed. □

Theorem 3.2 *Let H_1, H_2, A, C_i, Q_j and Γ be as in Theorem 3.1 and let F be an η-strongly monotone and $\tilde{L}$-Lipschitz continuous mapping on H_1. Let $\mu \in (0, 2\eta/\tilde{L}^2)$ be a fixed number and let conditions* (t), (γ), (ρ') *and* (λ) be satisfied. *Then, as $k \to \infty$, the sequence $\{x^k\}$, defined by* (1.16), T_k *in* (1.14) *and* τ_k *in* (1.15), *converges strongly to a point z_*, solving the variational inequality*:

$$z_* \in \Gamma : \ \langle Fz_*, z_* - z\rangle \le 0 \ \forall z \in \Gamma. \tag{3.10}$$

Proof. First, we also consider the case, when $P_1 = P_{C_p}\cdots P_{C_1}$ and $P_2 = P_{Q_r}\cdots P_{Q_1}$.

Set $z^k := (1-\gamma_k)x^k + \gamma_k T_k x^k$. By the similar argument as in the proof for

(3.4), taking a fixed point $z \in \Gamma$, we have that

$$\begin{aligned} \|z^k - z\|^2 \leq \|x^k - z\|^2 - 2\gamma_k\left(\frac{p+1}{2p} - \gamma_k\right)\|(I-P_1)x^k\|^2 \\ - 2\gamma_k\rho_k\left(\frac{r+1}{r} - \rho_k\right)\frac{f^2(x^k)}{(\|u^k\| + \lambda_k)^2}. \end{aligned} \tag{3.11}$$

First, we prove that $\{x^k\}$ is bounded. Indeed, from (1.16), Lemma 2.4, (3.11) and condition (ρ'), it follows that

$$\begin{aligned} \|x^{k+1} - z\| &= \|(I - t_k\mu F)z^k - (I - t_k\mu F)z - t_k\mu F z\| \\ &\leq (1 - t_k\tau)\|z^k - z\| + t_k\mu\|Fz\| \\ &\leq (1 - t_k\tau)\|x^k - z\| + t_k\mu\|Fz\| \\ &\leq \max\{\|x^1 - z\|, \mu\|Fz\|/\tau\}. \end{aligned}$$

Therefore, $\{x^k\}$ is bounded. Next, using Lemma 2.4 and (3.11) again, we obtain that

$$\begin{aligned} \|x^{k+1} - z\|^2 &= \|(I - t_k\mu F)z^k - (I - t_k\mu F)z - t_k\mu F z\|^2 \\ &\leq (1 - t_k\tau)\|z^k - z\|^2 - 2t_k\mu\langle Fz, x^{k+1} - z\rangle \\ &\leq (1 - t_k\tau)\|x^k - z\|^2 + 2t_k\mu\langle Fz, z - x^{k+1}\rangle \\ &\quad - 2\gamma_k\left(\frac{p+1}{2p} - \gamma_k\right)\|(I-P_1)x^k\|^2 \\ &\quad - 2\gamma_k\rho_k\left(\frac{r+1}{r} - \rho_k\right)\frac{f^2(x^k)}{(\|u^k\| + \lambda_k)^2}. \end{aligned} \tag{3.12}$$

Obviously, there exist two positive constant $\tilde{p}$ and $\tilde{r}$ such that, for all $k \in \mathbb{N}_+$,

$$2\gamma_k\left(\frac{p+1}{2p} - \gamma_k\right) \geq \tilde{p} \quad \text{and} \quad 2\gamma_k\rho_k\left(\frac{r+1}{r} - \rho_k\right) \geq \tilde{r}.$$

Thus, from (3.12) we get that

$$\begin{aligned} \|x^{k+1} - z\|^2 \leq (1 - t_k\tau)\|x^k - z\|^2 + 2t_k\mu\langle Fz, z - x^{k+1}\rangle \\ - \tilde{p}\|(I-P_1)x^k\|^2 - \tilde{r}\frac{f^2(x^k)}{(\|u^k\| + \lambda_k)^2}. \end{aligned} \tag{3.13}$$

We need only discuss two cases.

Case 1. There exists an integer $k_0 \geq 1$ such that $\|x^{k+1} - z\| \leq \|x^k - z\|$ for all $k \geq k_0$.

Then, $\lim_{k\to\infty}\|x^k - z\|$ exists. From (3.13), we can write that

$$\|x^{k+1}-z\|^2-\|x^k-z\|^2+t_k\tau\|x^k-z\|^2+\tilde{p}\|(I-P_1)x^k\|^2+\tilde{r}\frac{f^2(x^k)}{(\|u^k\|+\lambda_k)^2} \leq 2t_k\mu M_1, \tag{3.14}$$

where $M_1 \geq \|Fz\|\|z - x^{k+1}\|$. Since $\lim_{k\to\infty}\|x^k - z\|$ exists and $t_k \to 0$, letting k tend to infinity in (3.14), we get (3.6). By the similar argument as in the proof of Theorem 3.1, the sequence $\{x^k\}$ satisfies also (3.7), and hence, it converges weakly to $\tilde{z} \in \Gamma$. Moreover,

$$\limsup_{k\to\infty}\langle Fz_*, z_* - x^{k+1}\rangle = \lim_{l\to\infty}\langle Fz_*, z_* - x^{k_l+1}\rangle = \langle Fz_*, z_* - \tilde{z}\rangle \leq 0, \tag{3.15}$$

because $\tilde{z} \in \Gamma$ and z_* is the unique solution of (3.10). Now, from (3.13) with $\tilde{p} > 0$ and $\tilde{r} > 0$, we know that

$$\|x^{k+1} - z_*\|^2 \leq (1 - t_k\tau)\|x^k - z_*\|^2 + 2t_k\mu\langle Fz_*, z_* - x^{k+1}\rangle,$$

which together with Lemma 2.5 and (3.15) implies that $\|x^k - z_*\| \to 0$.

Case 2. There exists a subsequence $\{k_l\}$ of $\{k\}$ such that $\|x^{k_l}-z\| < \|x^{k_l+1}-z\|$ for all $l \in \mathbb{N}_+$.

Hence, by Lemma 2.6, there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}_+$ such that $m_k \to \infty$,

$$\|x^{m_k} - z\| \leq \|x^{m_k+1} - z\| \quad \text{and} \quad \|x^k - z\| \leq \|x^{m_k+1} - z\| \tag{3.16}$$

for each $k \in \mathbb{N}_+$. Next, according to (3.13) and the first inequality in (3.16),

$$\|x^{m_k} - z\|^2 \leq \frac{2\mu}{\tau}\langle Fz, z - x^{m_k+1}\rangle. \tag{3.17}$$

As in the proof of Theorem 3.1, the sequence $\{x^{m_k}\}$ has a weak cluster point in Γ and every weak convergent subsequence of $\{x^{m_k}\}$ converges weakly to an element in Γ. Therefore, we have

$$\limsup_{k\to\infty}\langle Fz_*, z_* - x^{m_k+1}\rangle \leq 0. \tag{3.18}$$

Using (3.17) with z replaced by z_* and (3.18), we get that

$$\lim_{k\to\infty}\|x^{m_k} - z_*\|^2 = 0. \tag{3.19}$$

Now, from (3.13) with z replaced by z_* and conditions (γ) with (τ), we can write that

$$\|x^{m_k+1} - z_*\|^2 \leq (1 - t_{m_k}\tau)\|x^{m_k} - z_*\|^2 + 2t_{m_k}\mu\langle Fz_*, z_* - x^{m_k+1}\rangle,$$

which together with (3.19) and $t_{m_k} \to 0$ implies that $\lim_{k\to\infty} \|x^{m_k+1} - z_*\|^2 = 0$. Then, using this fact and the second inequality in (3.16) with z replaced by z_*, we obtain that $\lim_{k\to\infty} \|x^k - z_*\| = 0$.

The other cases of P_1 and P_2 are considered similarly. This completes the proof. □

Remarks

It is easily seen that for a given contraction $h(x)$ with coefficient $\tilde{a} \in [0, 1)$, the mapping $F = I - h$ is $(1 - \tilde{a})$-strongly monotone and $(1 + \tilde{a})$-Lipschitz continuous.

1. Taking a fixed $\tilde{a} \in (0, 1)$ and $h = \tilde{a}I$, replacing F in (1.16) by $I - h = (1-\tilde{a})I$ and setting $t'_k := t_k\mu(1 - \tilde{a})$, we obtain a new method,

$$x^{k+1} = (1 - t'_k)[(1 - \gamma_k)I + \gamma_k T_k]x^k, \tag{3.20}$$

that converges strongly to an element in Γ under conditions (t), (γ), (ρ') and (λ). Thus, method (3.20) is an improvement modification of (1.14)-(1.15).

2. Setting $F = I - h$ in (1.16) with $h = \tilde{a}I + (1 - \tilde{a})u$ for a fixed point $u \in H_1$, we get a modified Krasnosel'skii-Mann-Halpern method

$$x^{k+1} = t'_k u + \beta'_k x^k + \gamma'_k T_k x^k, \tag{3.21}$$

that converges strongly under conditions in remark 1, where $\beta'_k = (1-\gamma_k)(1-t'_k)$ and $\gamma'_k = \gamma_k(1 - t'_k)$.

4. Numerical Example

Obviously, if $u = 0$ then (3.20) and (3.21) are coincided and in finite dimensional Hilbert spaces weak convergence is equivalent to strong one. So, in this section, we give an example in finite-dimensional Hilbert spaces for illustrating (3.20), and hence, (3.21) when $u = 0$.

We consider MSSFP (1.1) with $C = \cap_{i\in J_1} C_i$ and $Q = \cap_{j\in J_2} Q_j$ where

$$C_i = \{x \in \mathbb{E}^n : a^i_1 x_1 + a^i_2 x_2 + \cdots + a^i_n \le b_i\},$$

$a^i_j, b_i \in (-\infty; +\infty)$, for $i \in J_1$ and $1 \le j \le n$,

$$Q_j = \left\{ y \in \mathbb{E}^m : \sum_{l=1}^{m} (y_l - a^j_l)^2 \le r^2_j \right\},$$

$a^j_l, r_j \in (-\infty; +\infty)$, for $1 \le l \le m$ and $j \in J_2$, and A is an $m \times n$-matrix.

For computation, we consider the case: $n = 2$ and $m = 3$; $J_1 = \{1, 2, \cdots, 100\}$ and $J_2 = \{1, 2, \cdots, 200\}$; $a_1^i = 1/i, a_2^i = -1$ and $b_i = 0$ for all $i \in J_1$;

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix};$$

$a^j = (1/(j+1), 1/(j+1), 1/(j+1))$ and $r_j = 1$ for all $j \in J_2$. Clearly, $z_* = (0; 0)$ is the unique minimum norm solution.

We use the following values:

$$x^1 = (-2.0; -2.0);\ \rho_k = 0.5 + (1/2k);\ t_k = 0.25/k;\ \gamma_k = 0.2 + 1/(5k);$$
$$\alpha_i = 1/100;\ \beta_j = 1/200;\ \lambda_k = 0.01 + 1/k.$$

The computational results by algorithm (3.20) with τ_k defined by (1.15) and different forms of P_1 and P_2, are presented in 4 following numerical tables.

- $P_1 = P_{C_{100}} P_{C_{99}} \cdots P_{C_1}$ and $P_2 = P_{Q_{200}} P_{Q_{199}} \cdots P_{Q_1}$.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
100	-0.2524291284	0.1836867829	400	-0.1786197651	0.1299774326
200	-0.2123661475	0.1545338872	500	-0.1689360597	0.1229308263
300	-0.1919244414	0.1396589349	600	-0.1614137832	0.1174570413

- $P_1 = P_{C_{100}} P_{C_{99}} \cdots P_{C_1}$ and $P_2 = \sum_{j=1}^{200} \beta_j P_{Q_j}$.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
100	-0.4304284513	0.0688768475	400	-0.3044838988	0.0488304993
200	-0.3620512910	0.0579920951	500	-0.2879173003	0.0462031838
300	-0.3271601415	0.0524411032	600	-0.2750761647	0.0441623

- $P_1 = \sum_{i=1}^{100} \alpha_i P_{C_i}$ and $P_2 = \sum_{j=1}^{200} \beta_j P_{Q_j}$.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
100	-0.1948020988	0.0961874607	400	-0.1378411364	0.0680637848
200	-0.1638832695	0.0809229805	500	-0.1303682066	0.0643737967
300	-0.1481083747	0.0731335246	600	-0.1245632547	0.0615073757

- $P_1 = \sum_{i=1}^{100} P_{C_i}$ and $P_2 = P_{Q_{200}} P_{Q_{199}} \cdots P_{Q_1}$.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
100	-0.4082587527	-0.0045589461	400	-0.2886880085	-0.0028816231
200	-0.3433546286	-0.0035604815	500	-0.2729974061	-0.0026927938
300	-0.3102430999	-0.0031419744	600	-0.2608084207	-0.0025454572

Analyzing the computational results, we see that at the 600^{th} step, $|x_1| + |x_2| \approx 0.278; 0.319; 0.185$ and 0.262 for the first, second, third and fourth cases, respectively. So, the first case is the best one, because it is theoretically simpler than the third one.

5. Conclusions

We have proposed some new iterative methods with a self-adaptive step size for solving the multiple-sets split feasibility problem. We have also showed that some special cases of our methods are modifications of the Krasnoselskii-Mann and Halpern type ones. Numerical examples have been done for illustration.

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References

[1] Buong Ng., *Iterative algorithms for the multiple-sets split feasibility problem in Hilbert spaces,* Numer. Algorithms(2017), DOI 10.1007/s11075-017-0282-4.

[2] Byrne C., *A unified treatment of some iterative methods in signal processing and image reconstruction* Inverse Problems **20**(2004), 103-120.

[3] Byrne Ch. and Moudafi A., *Extensions of the CQ algorithm for the split feasibility and split equality problems,* Working paper UAG (2013).

[4] Censor Y. and Elfving T., *A multiprojection algorithm using Bregman projections in a product spaces* Numer. Algorithms **8**(1994), 221-239.

[5] Censor Y., Elfving T., Knop N. and Bortfeld T. *The multiple-sets split feasibility problem and its applications for inverse problems* Inverse Problems **21**(2005), 2071-2084.

[6] Censor Y Bortfeld T Martin B and Trofimov A 2006 A unified approach for inverse problems in intensity-modulated radiation therapy Phys. Med. Biol. **51** 2353-2365

[7] Censor Y Motova A and Segal A 2007 Perturbed projections and subgradient projections for the multiple-sets split feasibility problem J. Math. Anal. Appl. **327** 1244-1256

[8] Censor Y and Segal A 2008 Iterative projection methods in biomedical inverse problems Mathematical Methods in Biomedical Imaging and Intensity-Modulated Radiation Therapy (IMRT) ed Y Censor, M Jiang and A K Louis (Edizioni della Normale, Pisa, Italy) 65-96

[9] Chen R Li J and Ren Y 2013 Regularization method for the approximate split equality problem in infinite-dimensional Hilbert spaces Abstr. Appl. Anal. 813635

[10] Chen Y Guo Y Yu Y and Chen R 2012 Self-adaptive and relaxed self-adaptive projection methods for solving the multiple-set split feasibility problems Abstr. Appl. Anal. **2012** 958040

[11] Chuang ChSh and Du WSh 2016 Hybrid simultaneous algorithms for the split equality problem and applications Journal of Ineq. Appl. **2016** 198

[12] Combettes PL and Yamada I 2015 Compositions and convex combinations of averaged nonexpansive operators J. Math. Anal. Appl. **425** 55-70

[13] Dang Y Gao Y 2011 The strong convergence of a KM-CQ-like algorithm for a split feasibility problem Inverse Problems **27** 015007

[14] Dong QL He S and Yan HB 2013 Several projection algorithms for the split equality problem Wseas Transection on Mathematics **12**(11) 1087-1096

[15] Dong QL and He S 2014 Modified projection algorithms for solving the split equality problems The Scientific World Journal **2014** 328787

[16] Dong QL He S and Zhao J 2015 Solving the split equality problem without prior knowledge of operator norms Optimization **64**(9) 1887-1906

[17] Goebel K Kirk WA 1990 Topics in Metric Fixed Point Theory vol 28 Cambridge University Press Cambridge UK

[18] Halpern B 1967 Fixed points of nonexpanding mapps Bull. Am. Math. Soc. **73** 957-961

[19] He B Yang H Meng Q and Han D 2002 Modified Goldstein-Levitin-Polyak projection method for asymmetric strong monotone variational inequalities J. Optim. Theory Appl. **2** 129-143

[20] He H Liu S and Noor MA 2010 Some Krasnosel'skii-Mann algorithms and the multiple-sets split feasibility problem Fixed Point Theory and Applications **2010** 513956

[21] Li J and Chen R 2015 Several iterative algorithms for the multiple-sets split equality problem U.P.S. Sci.Bull. Seria A **17**(1) 87-98

[22] López G Marquez VM Wang F and Xu HK 2012 Solving the split feasibility problem without prior knowledge of matrix norms Inverse Problems **28** 085004

[23] Maingé PE 2008 Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization Set-Valued Var. Anal. **16** 899-912

[24] Mann WR 1953 Mean value method in iteration Proc. A.M.S. **4** 506-510

[25] Qu B Wang Ch and Xiu N 1917 Analysis on Newton projection method for the split feasibility problem Comput. Optim. Appl. **67** 175-199

[26] Schopfer F Schuster T and Luis AK 2008 An iterative regularization method for the solution of the split feasibility problem in Banach spaces Inverse Problem **24**(5) 055008

[27] Shehu Y 2016 Strong convergence theorem for multiple sets split feasibility problem in Banach spaces Numer. Func. Anal. Optim. **37**(8) 1021-1036

[28] Shi L Chen R and Wu YJ 2014 An iterative algorithm for the split equality and multiple-sets split equality problems Abstr. Appl. Anal. **2014** 620813

[29] Solodov MV Svaiter BF 2000 Forcing strong convergence of proximal point iterations in Hilbert spaces Math. Program. **87** 189-202

[30] Tang Y Zhu Ch and Yu H 2015 Iterative for solving the multiple-sets split feasibility problem with splitting self-adaptive step size Fixed Point Theory and Applications **2015**, 178

[31] Tian D Shi L and Chen R 2016 Iterative algorithm for solving the multiple-sets split equality problem with split self-adaptive step size in Hilbert spaces Journal of Ineq. Appl. **2016** 34

[32] Vuong PT Strodiot JJ and Hien NgV 2015 A gradient projection method for solving split equality and split feasibility problems in Hilbert spaces Optimization **64**(11) 2321-2341

[33] Wang Y Kim TH Fang X and He H 2017 The split common fixed point problem for demicontractive mappings and quasi-nonexpasnive mappings J. Nonlinear Sci. Appl. **10** 2976-2985

[34] Wang F 2017 A new method for split common fixed-point problem without priori knowledge of operator norm J. Fixed Point Theory Appl. DOI 10.1007/s11784-017-0434-0

[35] Wen M Peng J and Tang Y 2015 A cyclic and simultaneous iterative method for solving the multiple-sets split feasibility problem J. Optim. Theory Appl. **166** 844-860

[36] Xu HK 2003 An iterative approach to quadratic optimization J. Optim. Theory Appl. **116** 659-678

[37] Xu HK 2006 A variable Krasnosel'skii-Mann algorithm and multiple set split feasibility problem Inverse Problems **22** 2021-2034

[38] Xu HK 2011 Averaged mappings and the gradient-projection algorithm J. Optim. Theory Appl. **150** 360-378

[39] Yamada I 2001 The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings In Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications. (D.Butnariu, Y. Censor and S. Reich, Eds) North-Holland, Amsterdam 473-504

[40] Yao Y Postolache M and Liou YCh 2013 Strong convergence of a self-adaptive method for the split feasibility problems Fixed Point Theory and Applications **2013** 201

[41] Yu Y Yao Zh Wang Y and Fang X 2017 Applications of Mann's method for to the split common fixed point problem J Nonlinear Sci. Appl. **10** 1195-1200

[42] Zhang W Han D and Li Zh 2009 A self-adaptive projection method for solving the multiple-sets split feasibility problem Inverse Problems **25** 115001

[43] Zhao J Zhang Y and Yang Q 2012 Modified projection methods for the split feasibility problem and the the multiple-sets split feasibility problem Appl. Math. Comput. **219** 1644-1653

[44] Zhao J and Yang Q 2011 Self-adaptive projection methods for the multiple-sets split feasibility problems Inverse Problem **27** 035009