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ADDITIVITY OF JORDAN n-TUPLE DERIVABLE MAPS ON ALTERNATIVE RINGS

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Abstract

Let $\mathfrak{R}$ be an alternative ring. We study the additivity of maps $\delta : \mathfrak{R} \to \mathfrak{R}$ satisfying the following condition $\delta(a_n \circ (\cdots (a_2 \circ a_1) \cdots)) = \sum_{k=1}^{n} a_n \circ (\cdots (\delta(a_k) \circ (\cdots (a_2 \circ a_1) \cdots)) \cdots)$ for all $a_1, \cdots, a_n \in \mathfrak{R}$, where $a \circ b = ab + ba$ is the Jordan product of a and b in $\mathfrak{R}$. We prove that if $\mathfrak{R}$ contains a non-trivial idempotent satisfying some conditions, then δ is additive.

1 Introduction

A ring $\mathfrak{R}$ not necessarily associative or commutative is called *alternative* if $(xy)y = xy^2$ and $y^2x = y(yx)$, for all $x, y \in \mathfrak{R}$. Any associative ring is alternative, but the converse is not true.

An alternative ring $\mathfrak{R}$ is called *k-torsion free* if $k\,x = 0$ implies $x = 0$, for any $x \in \mathfrak{R}$, where k is a positive integer, and *prime* if $a\mathfrak{R}b = 0$ implies either $a = 0$ or $b = 0$. We consider the following convention for its multiplication operation: that $a_1a_2\cdots a_n = (\cdots(a_1a_2)\cdots)a_n$ and $a_n\cdots a_2a_1 = a_n(\cdots(a_2a_1)\cdots)$ and define the *Jordan product* $a \circ b$ of elements a, b in $\mathfrak{R}$ as $a \circ b = ab + ba$. A

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nonzero element $e_1 \in \mathfrak{R}$ is called an *idempotent* if $e_1e_1 = e_1$ and a *non-trivial idempotent* if it is an idempotent different from the multiplicative identity of $\mathfrak{R}$. Let us consider $\mathfrak{R}$ an alternative ring and fix a non-trivial idempotent $e_1 \in \mathfrak{R}$. Let $e_2 : \mathfrak{R} \to \mathfrak{R}$ and $e_2' : \mathfrak{R} \to \mathfrak{R}$ be linear operators given by $e_2(a) = a - e_1a$ and $e_2'(a) = a - ae_1$. Clearly $e_2^2 = e_2$, $(e_2')^2 = e_2'$ and we note that if $\mathfrak{R}$ has a unity, then we can consider $e_2 = 1 - e_1 \in \mathfrak{R}$. Let us denote $e_2(a)$ by e_2a and $e_2'(a)$ by ae_2. Then $\mathfrak{R}$ has a Peirce decomposition $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$, where $\mathfrak{R}_{ij} = e_i\mathfrak{R}e_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: (i) $\mathfrak{R}_{ij}\mathfrak{R}_{jl} \subseteq \mathfrak{R}_{il}$; (ii) $\mathfrak{R}_{ij}\mathfrak{R}_{ij} \subseteq \mathfrak{R}_{ji}$ $(i \neq j)$; (iii) $\mathfrak{R}_{ij}\mathfrak{R}_{kl} = 0$ if $j \neq k$ and $(i, j) \neq (k, l)$ and (iv) $x_{ij}^2 = 0$ $(i \neq j)$.

Let $\mathfrak{R}$ be an alternative ring and $\delta : \mathfrak{R} \to \mathfrak{R}$ be a map. We call δ a *Jordan derivable map* or *multiplicative Jordan derivation* if $\delta(a \circ b) = \delta(a) \circ b + a \circ \delta(b)$ for all $a, b \in \mathfrak{R}$ and a *Jordan n-tuple derivable map* or *multiplicative Jordan n-tuple derivation* if

$$\delta(a_n \circ (\cdots (a_2 \circ a_1) \cdots)) = \sum_{k=1}^{n} a_n \circ (\cdots (\delta(a_k) \circ (\cdots (a_2 \circ a_1) \cdots)) \cdots)$$

for all $a_1, \cdots, a_n \in \mathfrak{R}$.

It is easy to verify that every Jordan n-tuple derivable map on $\mathfrak{R}$ is also a Jordan $(pn - p + 1)$-tuple derivable map for each integer $p = 2, 3, \cdots$.

The authors in [2] studied the additivity of Jordan derivable maps defined on rings having at least one non-trivial idempotent element. They proved the following main theorem.

Theorem 1.1. *[2, Theorem 2.8.] Let $\mathfrak{R}$ be a ring with a nontrivial idempotent and satisfy*

(P1) If $a_{ij}x_{jk} = 0$ for all $x_{jk} \in \mathfrak{R}_{jk}$, then $a_{ij} = 0$;

(P2) If $x_{ij}a_{jk} = 0$ for all $x_{ij} \in \mathfrak{R}_{ij}$, then $a_{jk} = 0$;

(P3) If $a_{ii}x_{ii} + x_{ii}a_{ii} = 0$ for all $x_{ii} \in \mathfrak{R}_{ii}$, then $a_{ii} = 0$,

for $i, j, k \in \{1, 2\}$. If a mapping $\delta : \mathfrak{R} \to \mathfrak{R}$ satisfies

$$\delta(ab + ba) = \delta(a)b + a\delta(b) + \delta(b)a + b\delta(a)$$

for all $a, b \in \mathfrak{R}$, then δ is additive.

In addition, if $\mathfrak{R}$ is 2-torsion free, then δ is a Jordan derivation.

The authors in [1] generalized the Theorem 1.1 for the class of Jordan n-tuple derivable maps. They proved the following main theorem.

Theorem 1.2. *[1, Theorem 2.1.] Let $\mathfrak{R}$ be a ring 2 and $(2^{n-1} - 1)$-torsion free containing a non-trivial idempotent e_1 and satisfying the following conditions:*

(i) $e_i a e_j \mathfrak{R} e_k = 0$ *or* $e_k \mathfrak{R} e_i a e_j = 0$ *implies* $e_i a e_j = 0$ $(1 \leq i, j, k \leq 2)$;

(ii) $\underbrace{r_{22} \circ (\cdots (r_{22}}_{(n-1)-times} \circ a_{22}) \cdots) = 0$ *for all* $r_{22} \in \mathfrak{R}_{22}$, *implies* $a_{22} = 0$.

Then every Jordan n-tuple derivable map $\delta : \mathfrak{R} \to \mathfrak{R}$ is additive.

The aim of this article is to generalize the Theorem 1.2 to the class of Jordan n-tuple derivable maps on alternative rings.

2 The main result

Our main result reads as follows.

Theorem 2.1. *Let $\mathfrak{R}$ be an alternative ring 2 and $(2^{n-1} - 1)$-torsion free containing a non-trivial idempotent e_1 and satisfying the following conditions:*

(i) $e_i a e_j \mathfrak{R} e_k = 0$ *or* $e_k \mathfrak{R} e_i a e_j = 0$ *implies* $e_i a e_j = 0$ $(1 \leq i, j, k \leq 2)$;

(ii) $\underbrace{r_{22} \circ (\cdots (r_{22}}_{(n-1)-times} \circ a_{22}) \cdots) = 0$ *for all* $r_{22} \in \mathfrak{R}_{22}$, *implies* $a_{22} = 0$.

Then every Jordan n-tuple derivable map $\delta : \mathfrak{R} \to \mathfrak{R}$ is additive.

Based on the techniques used by Ferreira et al. [1] and Jing and Lu [2], we shall organize the proof of Theorem 2.1 in a series of lemmas. We observe that many formulas and relations presented in this paper are based in the mathematical induction principle. For sake of clarity of the whole text we have omitted their proofs.

The following three lemmas will be used throughout this paper whose proofs are elementary and therefore omitted.

Lemma 2.1. *Let $a = a_{11} + a_{12} + a_{21} + a_{22} \in \mathfrak{R}$. Then:*

(i) $r_{ij} a_{jk} = 0$ *for all* $r_{ij} \in \mathfrak{R}_{ij}$ $(1 \leq i, j, k \leq 2)$ *implies* $a_{jk} = 0$. *Dually,* $a_{jk} s_{kl} = 0$ *for all* $s_{kl} \in \mathfrak{R}_{kl}$ $(1 \leq j, k, l \leq 2)$ *implies* $a_{jk} = 0$*;*

(ii) $\underbrace{r_{ii} \circ (\cdots (r_{ii}}_{(n-1)-times} \circ a_{ii}) \cdots) = 0$ *for all* $r_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$, *implies* $a_{ii} = 0$.

Lemma 2.2. $\delta(0) = 0$.

Lemma 2.3. *For arbitrary elements $a_{11} \in \mathfrak{R}_{11}$, $b_{12} \in \mathfrak{R}_{12}$, $c_{21} \in \mathfrak{R}_{21}$ and $d_{22} \in \mathfrak{R}_{22}$ the following holds*

$$\delta(r^{pn-p}_{i_{pn-p} j_{pn-p}} \circ (\cdots (r^1_{i_1 j_1} \circ (a_{11} + b_{12} + c_{21} + d_{22})) \cdots))$$

$$
\begin{aligned}
= \quad & r^{pn-p}_{i_{pn-p}j_{pn-p}} \circ (\cdots (r^1_{i_1j_1} \circ \delta(a_{11}+b_{12}+c_{21}+d_{22}))\cdots) \\
& + \sum_{k=1}^{pn-p} r^{pn-p}_{i_{pn-p}j_{pn-p}} \circ (\cdots(\delta(r^k_{i_kj_k}) \circ (\cdots (r^1_{i_1j_1} \circ a_{11})\cdots))\cdots) \\
& + \sum_{k=1}^{pn-p} r^{pn-p}_{i_{pn-p}j_{pn-p}} \circ (\cdots(\delta(r^k_{i_kj_k}) \circ (\cdots (r^1_{i_1j_1} \circ b_{12})\cdots))\cdots) \\
& + \sum_{k=1}^{pn-p} r^{pn-p}_{i_{pn-p}j_{pn-p}} \circ (\cdots(\delta(r^k_{i_kj_k}) \circ (\cdots (r^1_{i_1j_1} \circ c_{21})\cdots))\cdots) \\
& + \sum_{k=1}^{pn-p} r^{pn-p}_{i_{pn-p}j_{pn-p}} \circ (\cdots(\delta(r^k_{i_kj_k}) \circ (\cdots (r^1_{i_1j_1} \circ d_{22})\cdots))\cdots) \qquad (1)
\end{aligned}
$$

for all $r^k_{i_kj_k} \in \mathfrak{R}_{i_kj_k}$ $(i_k, j_k = 1, 2; k = 1, \cdots, pn-p)$ *and for each integer* $p = 1, 2, \cdots$.

Lemma 2.4. *For arbitrary elements* $a_{11} \in \mathfrak{R}_{11}$, $b_{12} \in \mathfrak{R}_{12}$, $c_{21} \in \mathfrak{R}_{21}$ *and* $d_{22} \in \mathfrak{R}_{22}$ *the following hold: (i)* $\delta(a_{11}+b_{12}) = \delta(a_{11})+\delta(b_{12})$*; (ii)* $\delta(a_{11}+c_{21}) = \delta(a_{11}) + \delta(c_{21})$*; (iii)* $\delta(b_{12}+d_{22}) = \delta(b_{12}) + \delta(d_{22})$ *and (iv)* $\delta(c_{21}+d_{22}) = \delta(c_{21}) + \delta(d_{22})$.

Proof. For arbitrary elements $r^k_{i_kj_k} \in \mathfrak{R}_{i_kj_k}$ $(k = 1, \cdots, n-1)$, with $(i_1, j_1) = (1, 1)$ and $(i_k, j_k) = (2, 2)$ $(k = 2, \cdots, n-1)$, or $(i_k, j_k) = (2, 2)$ $(k = 1, \cdots, n-1)$, we have

$$
\begin{aligned}
& \delta(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ (a_{11}+b_{12}))\cdots)) \\
= \quad & \delta(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ a_{11})\cdots)) \\
& \qquad\qquad + \delta(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ b_{12})\cdots)) \\
= \quad & r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ \delta(a_{11}))\cdots) \\
& \qquad\qquad + r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ \delta(b_{12}))\cdots) \\
& + \sum_{k=1}^{n-1} r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots(\delta(r^k_{i_kj_k}) \circ (\cdots (r^1_{i_1j_1} \circ a_{11})\cdots))\cdots) \\
& + \sum_{k=1}^{n-1} r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots(\delta(r^k_{i_kj_k}) \circ (\cdots (r^1_{i_1j_1} \circ b_{12})\cdots))\cdots). \qquad (2)
\end{aligned}
$$

Taking into account $c_{21} = d_{22} = 0$ and $p = 1$, in (1), and subtracting (1) from (2), we obtain

$$
r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ (\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12})))\cdots) = 0. \qquad (3)
$$

It follows that, if $(i_1, j_1) = (1,1)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, then from (3) we obtain

$$(r_{11}^1(\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12})))r_{22}^2 \cdots r_{22}^{n-1} + r_{22}^{n-1} \cdots r_{22}^2((\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12}))r_{11}^1) = 0,$$

by multiplication table of the $\mathfrak{R}_{ij}$, which implies

$$(r_{11}^1(\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12})))r_{22}^2 \cdots r_{22}^{n-1} = r_{22}^{n-1} \cdots r_{22}^2((\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12}))r_{11}^1) = 0,$$

by the directness of the Peirce decomposition. Thus $(\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12}))_{12} = (\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12}))_{21} = 0$, by Lema 2.1(i). If $(i_k, j_k) = (2,2)$ and $r_{i_kj_k}^k = r_{22}^1$ for all $(k = 1, \cdots, n-1)$, then from (3), we have

$$r_{22}^1 \circ (\cdots (r_{22}^1 \circ (\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12}))) \cdots) = 0$$

which results $(\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12}))_{22} = 0$, by Lema 2.1(ii). Now, for arbitrary elements $r_{i_kj_k}^k \in \mathfrak{R}_{i_kj_k}$ $(k = 1, \cdots, 2n-2)$, $(i_1, j_1) = (2,1)$, $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$ and $(i_k, j_k) = (1,1)$ $(k = n, \cdots, 2n-2)$, we have

$$\begin{aligned}
&\delta(r_{i_{2n-2}j_{2n-2}}^{2n-2} \circ (\cdots (r_{i_1j_1}^1 \circ (a_{11}+b_{12})) \cdots)) \\
= {}& \delta(r_{i_{2n-2}j_{2n-2}}^{2n-2} \circ (\cdots (r_{i_1j_1}^1 \circ a_{11}) \cdots)) \\
&\qquad + \delta(r_{i_{2n-2}j_{2n-2}}^{2n-2} \circ (\cdots (r_{i_1j_1}^1 \circ b_{12}) \cdots)) \\
= {}& r_{i_{2n-2}j_{2n-2}}^{2n-2} \circ (\cdots (r_{i_1j_1}^1 \circ \delta(a_{11})) \cdots) \\
&\qquad + r_{i_{2n-2}j_{2n-2}}^{2n-2} \circ (\cdots (r_{i_1j_1}^1 \circ \delta(b_{12})) \cdots) \\
&+ \sum_{k=1}^{2n-2} r_{i_{2n-2}j_{2n-2}}^{2n-2} \circ (\cdots (\delta(r_{i_kj_k}^k) \circ (\cdots (r_{i_1j_1}^1 \circ a_{11}) \cdots)) \cdots) \\
&+ \sum_{k=1}^{2n-2} r_{i_{2n-2}j_{2n-2}}^{2n-2} \circ (\cdots (\delta(r_{i_kj_k}^k) \circ (\cdots (r_{i_1j_1}^1 \circ b_{12}) \cdots)) \cdots).
\end{aligned} \tag{4}$$

Taking into account $c_{21} = d_{22} = 0$ and $p = 2$, in (1), and subtracting (1) from (4), we obtain

$$r_{i_{2n-2}j_{2n-2}}^{2n-2} \circ (\cdots (r_{i_1j_1}^1 \circ (\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12}))) \cdots) = 0$$

which allows us to conclude that

$$(r_{22}^{n-1} \cdots r_{22}^2 r_{21}^1(\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12})))r_{11}^n \cdots r_{11}^{2n-2} = 0.$$

This yields $(\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12}))_{11} = 0$, by Lema 2.1(i). Consequently, $\delta(a_{11}+b_{12}) - \delta(a_{11}) - \delta(b_{12}) = 0$.

The other cases are proven similarly. □

Lemma 2.5. *For arbitrary elements $a_{12}, b_{12} \in \mathfrak{R}_{12}$, $b_{21}, c_{21} \in \mathfrak{R}_{21}$ and $t_{22} \in \mathfrak{R}_{22}$ the following hold: (i) $\delta(a_{12} + b_{12}t_{22}) = \delta(a_{12}) + \delta(b_{12}t_{22})$; (ii) $\delta(b_{21} + t_{22}c_{21}) = \delta(b_{21}) + \delta(t_{22}c_{21})$.*

Proof. First, we observe that the following identity is valid

$$\begin{aligned} & a_{12} + b_{12}t_{22} \\ = \; & \underbrace{e_1 \circ (\cdots (e_1}_{(n-2)-\text{times}} \circ ((a_{12} + t_{22}) \circ (e_1 + b_{12}))) \cdots). \end{aligned}$$

Hence, by Lemma 2.4 we have

$$\begin{aligned} & \delta(a_{12} + b_{12}t_{22}) \\ = \; & \delta(\underbrace{e_1 \circ (\cdots (e_1}_{(n-2)-\text{times}} \circ ((a_{12} + t_{22}) \circ (e_1 + b_{12}))) \cdots)) \\ = \; & e_1 \circ (\cdots (e_1 \circ (\cdots ((a_{12} + t_{22}) \circ \delta(e_1 + b_{12})) \cdots)) \cdots) \\ & + e_1 \circ (\cdots (e_1 \circ (\cdots (\delta(a_{12} + t_{22}) \circ (e_1 + b_{12})) \cdots)) \cdots) \\ & + \sum_{k=3}^{n-1} e_1 \circ (\cdots (\underbrace{\delta(e_1) \circ (\cdots ((a_{12} + t_{22}) \circ (e_1 + b_{12}))}_{k-\text{times}} \cdots)) \cdots) \\ = \; & e_1 \circ (\cdots (e_1 \circ (\cdots (a_{12} \circ \delta(e_1)) \cdots)) \cdots) \\ & + e_1 \circ (\cdots (e_1 \circ (\cdots (a_{12} \circ \delta(b_{12})) \cdots)) \cdots) \\ & + e_1 \circ (\cdots (e_1 \circ (\cdots (t_{22} \circ \delta(e_1)) \cdots)) \cdots) \\ & + e_1 \circ (\cdots (e_1 \circ (\cdots (t_{22} \circ \delta(b_{12})) \cdots)) \cdots) \\ & + e_1 \circ (\cdots (e_1 \circ (\cdots (\delta(a_{12}) \circ e_1) \cdots)) \cdots) \\ & + e_1 \circ (\cdots (e_1 \circ (\cdots (\delta(a_{12}) \circ b_{12}) \cdots)) \cdots) \\ & + e_1 \circ (\cdots (e_1 \circ (\cdots (\delta(t_{22}) \circ e_1) \cdots)) \cdots) \\ & + e_1 \circ (\cdots (e_1 \circ (\cdots (\delta(t_{22}) \circ b_{12}) \cdots)) \cdots) \\ & + \sum_{k=3}^{n-1} e_1 \circ (\cdots (\underbrace{\delta(e_1) \circ (\cdots (a_{12} \circ e_1)}_{k-\text{times}} \cdots)) \cdots) \\ & + \sum_{k=3}^{n-1} e_1 \circ (\cdots (\underbrace{\delta(e_1) \circ (\cdots (a_{12} \circ b_{12})}_{k-\text{times}} \cdots)) \cdots) \\ & + \sum_{k=3}^{n-1} e_1 \circ (\cdots (\underbrace{\delta(e_1) \circ (\cdots (t_{22} \circ e_1)}_{k-\text{times}} \cdots)) \cdots) \\ & + \sum_{k=3}^{n-1} e_1 \circ (\cdots (\underbrace{\delta(e_1) \circ (\cdots (t_{22} \circ b_{12})}_{k-\text{times}} \cdots)) \cdots) \end{aligned}$$

$$\begin{aligned} &= \delta(e_1 \circ (\cdots (e_1 \circ (a_{12} \circ e_1)) \cdots)) + \delta(e_1 \circ (\cdots (e_1 \circ (a_{12} \circ b_{12})) \cdots)) \\ &\quad + \delta(e_1 \circ (\cdots (e_1 \circ (t_{22} \circ e_1)) \cdots)) + \delta(e_1 \circ (\cdots (e_1 \circ (t_{22} \circ b_{12})) \cdots)) \\ &= \delta(a_{12}) + \delta(b_{12}t_{22}). \end{aligned}$$

Using a similar argument to the case (i), we prove that $\delta(b_{21} + t_{22}c_{21}) = \delta(b_{21}) + \delta(t_{22}c_{21})$, from the identity

$$\begin{aligned} & b_{21} + t_{22}c_{21} \\ = \; & \underbrace{e_1 \circ (\cdots (e_1}_{(n-2)-\text{times}} \circ ((c_{21} + e_1) \circ (b_{21} + t_{22}))) \cdots). \end{aligned}$$

□

Lemma 2.6. *For arbitrary elements $a_{12}, b_{12} \in \mathfrak{R}_{12}$ and $b_{21}, c_{21} \in \mathfrak{R}_{21}$ the following hold: (i) $\delta(a_{12} + b_{12}) = \delta(a_{12}) + \delta(b_{12})$; (ii) $\delta(b_{21} + c_{21}) = \delta(b_{21}) + \delta(c_{21})$.*

Proof. For arbitrary elements $r^k_{i_kj_k} \in \mathfrak{R}_{i_kj_k}$ $(k = 1, \cdots, n-1)$ with $(i_1, j_1) = (1, 1)$ and $(i_k, j_k) = (2, 2)$ $(k = 2, \cdots, n-1)$, or $(i_k, j_k) = (2, 2)$ $(k = 1, \cdots, n-1)$, or $(i_1, j_1) = (1, 2)$ and $(i_k, j_k) = (2, 2)$ $(k = 2, \cdots, n-1)$, we have by Lemma 2.5 that

$$\begin{aligned} & \delta(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ (a_{12} + b_{12})) \cdots)) \\ = \; & \delta(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ a_{12}) \cdots)) \\ & \qquad + \delta(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ b_{12}) \cdots)) \\ = \; & r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ \delta(a_{12})) \cdots) \\ & \qquad + r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ \delta(b_{12})) \cdots) \\ & + \sum_{k=1}^{n-1} r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (\delta(r^k_{i_kj_k}) \circ (\cdots (r^1_{i_1j_1} \circ a_{12}) \cdots)) \cdots) \\ & + \sum_{k=1}^{n-1} r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (\delta(r^k_{i_kj_k}) \circ (\cdots (r^1_{i_1j_1} \circ b_{12}) \cdots)) \cdots). \end{aligned} \tag{5}$$

Taking into account $a_{11} = c_{21} = d_{22} = 0$ and replacing b_{12} by $a_{12} + b_{12}$, in (1), and subtracting (1) from (5), it results

$$r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ (\delta(a_{12} + b_{12}) - \delta(a_{12}) - \delta(b_{12}))) \cdots) = 0. \tag{6}$$

Hence, if $(i_1, j_1) = (1, 1)$ and $(i_k, j_k) = (2, 2)$ $(k = 2, \cdots, n-1)$, from (6) we obtain

$$(r_{11}^1(\delta(a_{12}+b_{12})-\delta(a_{12})-\delta(b_{12})))r_{22}^2\cdots r_{22}^{n-1} + r_{22}^{n-1}\cdots r_{22}^2((\delta(a_{12}+b_{12})-\delta(a_{12})-\delta(b_{12}))r_{11}^1)=0,$$

which implies

$$(r_{11}^1(\delta(a_{11}+b_{12})-\delta(a_{11})-\delta(b_{12})))r_{22}^2\cdots r_{22}^{n-1} = r_{22}^{n-1}\cdots r_{22}^2((\delta(a_{11}+b_{12})-\delta(a_{11})-\delta(b_{12}))r_{11}^1)=0,$$

by directness of the Peirce decomposition. Thus

$$(\delta(a_{12}+b_{12})-\delta(a_{12})-\delta(b_{12}))_{12}=(\delta(a_{12}+b_{12})-\delta(a_{12})-\delta(b_{12}))_{21}=0.$$

If $(i_k,j_k)=(2,2)$ and $r_{i_kj_k}^k=r_{22}^1$ for all $(k=1,\cdots,n-1)$, then from (6) we get

$$r_{22}^1\circ(\cdots(r_{22}^1\circ(\delta(a_{12}+b_{12})-\delta(a_{12})-\delta(b_{12})))\cdots)=0.$$

which shows that $(\delta(a_{12}+b_{12})-\delta(a_{12})-\delta(b_{12}))_{22}=0$, by Lema 2.1(ii). If $(i_1,j_1)=(1,2)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$, then from (6) yet and the previous cases, we conclude that

$$(\delta(a_{12}+b_{12})-\delta(a_{12})-\delta(b_{12}))r_{12}^1r_{22}^2\cdots r_{22}^{n-1}=0.$$

This implies $(\delta(a_{12}+b_{12})-\delta(a_{12})-\delta(b_{12}))_{11}=0$, by Lema 2.1(i). Consequently, $\delta(a_{12}+b_{12})-\delta(a_{12})-\delta(b_{12})=0$.

The other case is proven similarly. □

Lemma 2.7. *For arbitrary elements $a_{11},b_{11}\in\mathfrak{R}_{11}$ and $c_{22},d_{22}\in\mathfrak{R}_{22}$ the following hold: (i) $\delta(a_{11}+b_{11})=\delta(a_{11})+\delta(b_{11})$; (ii) $\delta(c_{22}+d_{22})=\delta(c_{22})+\delta(d_{22})$.*

Proof. For arbitrary elements $r_{i_kj_k}^k\in\mathfrak{R}_{i_kj_k}$ $(k=1,\cdots,n-1)$ with $(i_1,j_1)=(1,1)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$, or $(i_k,j_k)=(2,2)$ $(k=1,\cdots,n-1)$, or $(i_1,j_1)=(1,2)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$, we have by Lemma 2.6(i) that

$$\begin{aligned}
&\delta(r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(r_{i_1j_1}^1\circ(a_{11}+b_{11}))\cdots))\\
=\;&\delta(r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(r_{i_1j_1}^1\circ a_{11})\cdots))\\
&\qquad+\delta(r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(r_{i_1j_1}^1\circ b_{11})\cdots))\\
=\;&r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(r_{i_1j_1}^1\circ\delta(a_{11}))\cdots)\\
&\qquad+r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(r_{i_1j_1}^1\circ\delta(b_{11}))\cdots)\\
&+\sum_{k=1}^{n-1}r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(\delta(r_{i_kj_k}^k)\circ(\cdots(r_{i_1j_1}^1\circ a_{11})\cdots))\cdots)
\end{aligned}$$

$$+\sum_{k=1}^{n-1} r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots(\delta(r^k_{i_kj_k}) \circ (\cdots(r^1_{i_1j_1} \circ b_{11})\cdots))\cdots). \qquad (7)$$

Taking into account $b_{12} = c_{21} = d_{22} = 0$ and replacing a_{11} by $a_{11}+b_{11}$, in (1), and subtracting (1) from (7), we obtain

$$r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots(r^1_{i_1j_1} \circ (\delta(a_{11} + b_{11}) - \delta(a_{11}) - \delta(b_{11})))\cdots) = 0. \quad (8)$$

Hence, if $(i_1, j_1) = (1, 1)$ and $(i_k, j_k) = (2, 2)$ $(k = 2, \cdots, n-1)$, from (8) we get

$$\begin{aligned}(r^1_{11}(\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11})))r^2_{22}\cdots r^{n-1}_{22}\\ = r^{n-1}_{22}\cdots r^2_{22}((\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}))r^1_{11}) = 0\end{aligned}$$

which shows that

$$(\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}))_{12} = (\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}))_{21} = 0.$$

If $(i_k, j_k) = (2, 2)$ and $r^k_{i_kj_k} = r^1_{22}$ for all $(k = 1, \cdots, n-1)$, then from (8) we obtain

$$r^1_{22} \circ (\cdots(r^1_{22} \circ (\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11})))\cdots) = 0$$

which results $(\delta(a_{11}+b_{11})-\delta(a_{11})-\delta(b_{11}))_{22} = 0$, by Lema 2.1(ii). If $(i_1, j_1) = (1, 2)$ and $(i_k, j_k) = (2, 2)$ $(k = 2, \cdots, n-1)$, then from (8) yet and the previous cases, we obtain

$$(\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}))r^1_{12}r^2_{22}\cdots r^{n-1}_{22} = 0,$$

which yields $(\delta(a_{11}+b_{11})-\delta(a_{11})-\delta(b_{11}))_{11} = 0$. Consequently, $\delta(a_{11}+b_{11})-\delta(a_{11}) - \delta(b_{11}) = 0$.

The other case is proven similarly. □

Lemma 2.8. *For arbitrary elements $b_{12} \in \mathfrak{R}_{12}$ and $c_{21} \in \mathfrak{R}_{21}$ the following holds $\delta(b_{12} + c_{21}) = \delta(b_{12}) + \delta(c_{21})$.*

Proof. For arbitrary elements $r^k_{i_kj_k} \in \mathfrak{R}_{i_kj_k}$ $(k = 1, \cdots, n-1)$ with $(i_k, j_k) = (1, 1)$ $(k = 1, \cdots, n-2)$ and $(i_{n-1}, j_{n-1}) = (1, 2)$, or $(i_k, j_k) = (2, 2)$ $(k = 1, \cdots, n-2)$ and $(i_{n-1}, j_{n-1}) = (2, 1)$, or $(i_1, j_1) = (1, 2)$ and $(i_k, j_k) = (1, 1)$ $(k = 2, \cdots, n-1)$, we have that

$$\begin{aligned}&\delta(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots(r^1_{i_1j_1} \circ (b_{12} + c_{21}))\cdots))\\ =\ &\delta(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots(r^1_{i_1j_1} \circ b_{12})\cdots))\\ &\qquad +\delta(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots(r^1_{i_1j_1} \circ c_{21})\cdots))\end{aligned}$$

$$\begin{aligned} = \quad & r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ \delta(b_{12})) \cdots) \\ & \qquad\qquad\qquad + r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ \delta(c_{21})) \cdots) \\ & + \sum_{k=1}^{n-1} r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (\delta(r^k_{i_kj_k}) \circ (\cdots (r^1_{i_1j_1} \circ b_{12}) \cdots)) \cdots) \\ & + \sum_{k=1}^{n-1} r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (\delta(r^k_{i_kj_k}) \circ (\cdots (r^1_{i_1j_1} \circ c_{21}) \cdots)) \cdots). \end{aligned} \tag{9}$$

Taking into account $a_{11} = d_{22} = 0$ in (1) and subtracting (1) from (9), we obtain

$$r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (r^1_{i_1j_1} \circ (\delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21}))) \cdots) = 0. \tag{10}$$

Hence, if $(i_k, j_k) = (1, 1)$ $(k = 1, \cdots, n-2)$ and $(i_{n-1}, j_{n-1}) = (1, 2)$, from (10) we get

$$r^{n-1}_{12} \circ (r^{n-2}_{11} \circ (\cdots (r^1_{11} \circ (\delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21}))) \cdots)) = 0. \tag{11}$$

Multiplying (11) from right by t_{11}, then

$$(r^{n-1}_{12}((\delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21}))r^1_{11} \cdots r^{n-2}_{11}))t_{11} = 0$$

which implies $\delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21}))_{21} = 0$. If $(i_k, j_k) = (2, 2)$ $(k = 1, \cdots, n-2)$ and $(i_{n-1}, j_{n-1}) = (2, 1)$, from (10) we have

$$r^{n-1}_{21} \circ (r^{n-2}_{22} \circ (\cdots (r^1_{22} \circ (\delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21}))) \cdots)) = 0. \tag{12}$$

Multiplying (12) from right by t_{22}, then

$$(r^{n-1}_{21}((\delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21}))r^1_{22} \cdots r^{n-2}_{22}))t_{22} = 0$$

which results $(\delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21}))_{12} = 0$. Now, let us prove that $(\delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21}))_{11} = 0$. In fact, first note that

$$b_{12} + c_{21} = \underbrace{e_1 \circ (\cdots (e_1}_{(n-1)-\text{times}} \circ (b_{12} + c_{21})) \cdots).$$

Hence

$$\begin{aligned} & \delta(b_{12} + c_{21}) \\ = \quad & \delta(e_1 \circ (\cdots (e_1 \circ (b_{12} + c_{21})) \cdots)) \\ = \quad & e_1 \circ (\cdots (e_1 \circ \delta(b_{12} + c_{21})) \cdots) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n-1} e_1 \circ (\cdots(\delta(e_1) \circ (\cdots(e_1 \circ (b_{12}+c_{21}))\cdots))\cdots) \\
= \quad & e_1 \circ (\cdots(e_1 \circ \delta(b_{12}+c_{21}))\cdots) \\
&+\sum_{k=1}^{n-1} e_1 \circ (\cdots(\delta(e_1) \circ (\cdots(e_1 \circ b_{12})\cdots))\cdots) \\
&+\sum_{k=1}^{n-1} e_1 \circ (\cdots(\delta(e_1) \circ (\cdots(e_1 \circ c_{21})\cdots))\cdots). \qquad (13)
\end{aligned}
$$

Also, note that

$$b_{12} = \underbrace{e_1 \circ (\cdots(e_1}_{(n-1)-\text{times}} \circ b_{12})\cdots)$$

which implies

$$
\begin{aligned}
& \delta(b_{12}) \\
= \quad & \delta(e_1 \circ (\cdots(e_1 \circ b_{12})\cdots)) \\
= \quad & e_1 \circ (\cdots(e_1 \circ \delta(b_{12}))\cdots) \\
&+\sum_{k=1}^{n-1} e_1 \circ (\cdots(\delta(e_1) \circ (\cdots(e_1 \circ b_{12})\cdots))\cdots). \qquad (14)
\end{aligned}
$$

Similarly, we have

$$c_{21} = \underbrace{e_1 \circ (\cdots(e_1}_{(n-1)-\text{times}} \circ c_{21})\cdots)$$

which results

$$
\begin{aligned}
& \delta(c_{21}) \\
= \quad & \delta(e_1 \circ (\cdots(e_1 \circ c_{21})\cdots)) \\
= \quad & e_1 \circ (\cdots(e_1 \circ \delta(c_{21}))\cdots) \\
&+\sum_{k=1}^{n-1} e_1 \circ (\cdots(\delta(e_1) \circ (\cdots(e_1 \circ c_{21})\cdots))\cdots). \qquad (15)
\end{aligned}
$$

Hence, subtracting (13) from (14) and (15), we obtain

$$
\begin{aligned}
& \delta(b_{12}+c_{21}) - \delta(b_{12}) - \delta(c_{21}) \\
= \quad & e_1 \circ (\cdots(e_1 \circ (\delta(b_{12}+c_{21}) - \delta(b_{12}) - \delta(c_{21})))\cdots). \qquad (16)
\end{aligned}
$$

which implies $(2^{n-1}-1)e_1(\delta(b_{12}+c_{21})-\delta(b_{12})-\delta(c_{21}))e_1 = 0$. Thus $(\delta(b_{12}+c_{21})-\delta(b_{12})-\delta(c_{21}))_{11} = 0$, since $\mathfrak{R}$ is a $(2^{n-1}-1)$-torsion free ring. Finally, if $(i_1, j_1) = (1, 2)$ and $(i_k, j_k) = (1, 1)$ $(k = 2, \cdots, n-1)$, from (10) yet, we have

$$r_{11}^{n-1} \circ (r_{11}^{n-2} \circ (\cdots (r_{12}^1 \circ (\delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21}))) \cdots)) = 0.$$

which yields

$$r_{11}^{n-1} \cdots r_{11}^2 r_{12}^1 (\delta(b_{12}+c_{21}) - \delta(b_{12}) - \delta(c_{21})) = 0, \tag{17}$$

by the previous cases. The identity (17) allows us to conclude that $(\delta(b_{12}+c_{21})-\delta(b_{12})-\delta(c_{21}))_{22} = 0$. Consequently, $\delta(b_{12}+c_{21})-\delta(b_{12})-\delta(c_{21}) = 0$. □

Lemma 2.9. *For arbitrary elements* $a_{11} \in \mathfrak{R}_{11}$, $b_{12} \in \mathfrak{R}_{12}$, $c_{21} \in \mathfrak{R}_{21}$ *and* $d_{22} \in \mathfrak{R}_{22}$ *the following hold: (i)* $\delta(a_{11} + b_{12} + c_{21}) = \delta(a_{11}) + \delta(b_{12}) + \delta(c_{21})$; *(ii)* $\delta(b_{12} + c_{21} + d_{22}) = \delta(b_{12}) + \delta(c_{21}) + \delta(d_{22})$.

Proof. For arbitrary elements $r_{i_k j_k}^k \in \mathfrak{R}_{i_k j_k}$ $(k = 1, \cdots, n-1)$ with $(i_1, j_1) = (1, 1)$ and $(i_k, j_k) = (2, 2)$ $(k = 2, \cdots, n-1)$, or $(i_k, j_k) = (2, 2)$ $(k = 1, \cdots, n-1)$, or $(i_1, j_1) = (1, 2)$ and $(i_k, j_k) = (2, 2)$ $(k = 2, \cdots, n-1)$, we have by Lemma 2.8 that

$$\begin{aligned}
& \delta(r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots (r_{i_1 j_1}^1 \circ (a_{11} + b_{12} + c_{21})) \cdots)) \\
= \; & \delta(r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots (r_{i_1 j_1}^1 \circ a_{11}) \cdots)) \\
& \qquad\qquad + \delta(r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots (r_{i_1 j_1}^1 \circ b_{12}) \cdots)) \\
& + \delta(r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots (r_{i_1 j_1}^1 \circ c_{21}) \cdots)) \\
= \; & r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots (r_{i_1 j_1}^1 \circ \delta(a_{11})) \cdots) \\
& \qquad\qquad + r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots (r_{i_1 j_1}^1 \circ \delta(b_{12})) \cdots) \\
& + r_{i_{n-1}j_{-1}}^{n-1} \circ (\cdots (r_{i_1 j_1}^1 \circ \delta(c_{21})) \cdots) \\
& + \sum_{k=1}^{n-1} r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots (\delta(r_{i_k j_k}^k) \circ (\cdots (r_{i_1 j_1}^1 \circ a_{11}) \cdots)) \cdots) \\
& + \sum_{k=1}^{n-1} r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots (\delta(r_{i_k j_k}^k) \circ (\cdots (r_{i_1 j_1}^1 \circ b_{12}) \cdots)) \cdots) \\
& + \sum_{k=1}^{n-1} r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots (\delta(r_{i_k j_k}^k) \circ (\cdots (r_{i_1 j_1}^1 \circ c_{21}) \cdots)) \cdots).
\end{aligned} \tag{18}$$

Taking into account $d_{22} = 0$ in (1) and subtracting (1) from (18), we conclude

$$r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots (r_{i_1 j_1}^1 \circ (\delta(a_{11} + b_{12} + c_{21}) - \delta(a_{11}) - \delta(b_{11}) - \delta(c_{21}))) \cdots)$$

$$= 0. \quad (19)$$

Hence, if $(i_1, j_1) = (1,1)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, from (19) we obtain

$$(r_{11}^1(\delta(a_{11} + b_{12} + c_{21}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21})))r_{22}^2 \cdots r_{22}^{n-1} + r_{22}^{n-1} \cdots r_{22}^2((\delta(a_{11} + b_{12} + c_{21}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21}))r_{11}^1) = 0,$$

which yields

$$(r_{11}^1(\delta(a_{11} + b_{12}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21})))r_{22}^2 \cdots r_{22}^{n-1} = r_{22}^{n-1} \cdots r_{22}^2((\delta(a_{11} + b_{12}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21}))r_{11}^1) = 0.$$

This results

$$(\delta(a_{11} + b_{12} + c_{21}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21}))_{12} = (\delta(a_{11} + b_{12} + c_{21}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21}))_{21} = 0.$$

If $(i_k, j_k) = (2,2)$ and $r_{i_k j_k}^k = r_{22}^1$ for all $(k = 1, \cdots, n-1)$, then from (19), we obtain

$$r_{22}^1 \circ (\cdots (r_{22}^1 \circ \delta(a_{11} + b_{12} + c_{21}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21}))) \cdots) = 0.$$

which yields $(\delta(a_{11} + b_{12} + c_{21}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21}))_{22} = 0$, by Lema 2.1(ii). If $(i_1, j_1) = (1,2)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, then from (19) yet and the previous cases, we conclude that

$$(\delta(a_{11} + b_{12} + c_{21}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21}))r_{12}^1 r_{22}^2 \cdots r_{22}^{n-1} = 0.$$

This implies $(\delta(a_{11} + b_{12} + c_{21}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21}))_{11} = 0$. Consequently, $\delta(a_{11} + b_{12} + c_{21}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21}) = 0$.

The other case is proven similarly. □

Lemma 2.10. *For arbitrary elements $a_{11} \in \mathfrak{R}_{11}$, $b_{12} \in \mathfrak{R}_{12}$, $c_{21} \in \mathfrak{R}_{21}$ and $d_{22} \in \mathfrak{R}_{22}$ holds $\delta(a_{11} + b_{12} + c_{21} + d_{22}) = \delta(a_{11}) + \delta(b_{12}) + \delta(c_{21}) + \delta(d_{22})$.*

Proof. For arbitrary elements $r_{i_k j_k}^k \in \mathfrak{R}_{i_k j_k}$ $(k = 1, \cdots, n-1)$ with $(i_1, j_1) = (1,1)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, or $(i_k, j_k) = (1,1)$ $(k = 1, \cdots, n-1)$, or $(i_k, j_k) = (2,2)$ $(k = 1, \cdots, n-1)$, we have by Lemma 2.9 that

$$\begin{aligned} & \delta(r_{i_{n-1} j_{n-1}}^{n-1} \circ (\cdots (r_{i_1 j_1}^1 \circ (a_{11} + b_{12} + c_{21} + d_{22})) \cdots)) \\ = \; & \delta(r_{i_{n-1} j_{n-1}}^{n-1} \circ (\cdots (r_{i_1 j_1}^1 \circ a_{11}) \cdots)) \\ & + \delta(r_{i_{n-1} j_{n-1}}^{n-1} \circ (\cdots (r_{i_1 j_1}^1 \circ b_{12}) \cdots)) \end{aligned}$$

$$
\begin{aligned}
&+\delta(r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(r^1_{i_1j_1}\circ c_{21})\cdots))\\
&\qquad\qquad\qquad +\delta(r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(r^1_{i_1j_1}\circ d_{22})\cdots))\\
=\;& r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(r^1_{i_1j_1}\circ\delta(a_{11}))\cdots)\\
&\qquad\qquad\qquad +r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(r^1_{i_1j_1}\circ\delta(b_{12}))\cdots)\\
&+r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(r^1_{i_1j_1}\circ\delta(c_{21}))\cdots)\\
&\qquad\qquad\qquad +r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(r^1_{i_1j_1}\circ\delta(d_{22}))\cdots)\\
&+\sum_{k=1}^{n-1} r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(\delta(r^k_{i_kj_k})\circ(\cdots(r^1_{i_1j_1}\circ a_{11})\cdots))\cdots)\\
&+\sum_{k=1}^{n-1} r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(\delta(r^k_{i_kj_k})\circ(\cdots(r^1_{i_1j_1}\circ b_{12})\cdots))\cdots)\\
&+\sum_{k=1}^{n-1} r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(\delta(r^k_{i_kj_k})\circ(\cdots(r^1_{i_1j_1}\circ c_{21})\cdots))\cdots)\\
&+\sum_{k=1}^{n-1} r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(\delta(r^k_{i_kj_k})\circ(\cdots(r^1_{i_1j_1}\circ d_{22})\cdots))\cdots). \qquad (20)
\end{aligned}
$$

Subtracting (1) from (20), it implies

$$
\begin{aligned}
& r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(r^1_{i_1j_1}\circ(\delta(a_{11}+b_{12}+c_{21}+d_{22})\\
&\qquad -\delta(a_{11})-\delta(b_{11})-\delta(c_{21})-\delta(d_{22})))\cdots)=0. \qquad (21)
\end{aligned}
$$

Hence, if $(i_1,j_1)=(1,1)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$, from (21) we obtain $(\delta(a_{11}+b_{12}+c_{21}+d_{22})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21})-\delta(d_{22}))_{12}=0$ and $(\delta(a_{11}+b_{12}+c_{21}+d_{22})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21})-\delta(d_{22}))_{21}=0$. If $(i_k,j_k)=(1,1)$ and $r^k_{i_kj_k}=r^1_{11}$ for all $(k=1,\cdots,n-1)$, then $(\delta(a_{11}+b_{12}+c_{21}+d_{22})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21})-\delta(d_{22}))_{11}=0$, from (21) and Lemma 2.1(ii). If $(i_k,j_k)=(2,2)$ and $r^k_{i_kj_k}=r^1_{22}$ for all $(k=1,\cdots,n-1)$, from (21) and Lemma 2.1(ii) again we have $(\delta(a_{11}+b_{12}+c_{21}+d_{22})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21})-\delta(d_{22}))_{22}=0$. Thus we conclude that $\delta(a_{11}+b_{12}+c_{21}+d_{22})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21})-\delta(d_{22})=0$. □

Now we are able to prove the Theorem 2.1. Our proof is similar those presented by Ferreira et al. [1] and Jing and Lu [2].

Proof of Theorem 2.1. Let $a=a_{11}+a_{12}+a_{21}+a_{22}$ and $b=b_{11}+b_{12}+b_{21}+b_{22}$ be two arbitrary elements of $\mathfrak{R}$. From Lemmas 2.6, 2.7 and 2.10 we have

$$
\begin{aligned}
&\delta(a+b)\\
=\;&\delta((a_{11}+b_{11})+(a_{12}+b_{12})+(a_{21}+b_{21})+(a_{22}+b_{22}))
\end{aligned}
$$

$$
\begin{aligned}
&= \delta(a_{11}+b_{11})+\delta(a_{12}+b_{12})+\delta(a_{21}+b_{21})+\delta(a_{22}+b_{22})\\
&= \delta(a_{11})+\delta(b_{11})+(a_{12})+\delta(b_{12})+\delta(a_{21})+\delta(b_{21})+\delta(a_{22})+\delta(b_{22})\\
&= \delta(a_{11}+a_{12}+a_{21}+a_{22})+\delta(b_{11}+b_{12}+b_{21}+b_{22})\\
&= \delta(a)+\delta(b).
\end{aligned}
$$

Thus, δ is additive. The prove is complete. □

Corollary 2.1. *Let $\mathfrak{R}$ be a prime alternative ring 2 and $(2^{n-1}-1)$-torsion free, containing a non-trivial idempotent e_1 and satisfying the following condition: $\underbrace{e_2re_2 \circ (\cdots (e_2re_2}_{(n-1)-times} \circ e_2ae_2)\cdots) = 0$, for all $r \in \mathfrak{R}$, implies $e_2ae_2 = 0$. Then every Jordan n-tuple derivable map $\delta : \mathfrak{R} \to \mathfrak{R}$ is additive.*

References

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