East-West J. of Mathematics: Vol. 20, No 1 (2018) pp. 37-43

ON THE SPINOR NORM ON UNITARY GROUPS

Ngo Van Dinh

Department of Mathematics and Informatics
Thai Nguyen University of Sciences
Thai Nguyen, Vietnam
e-mail: dinh.ngo@tnus.edu.vn

Abstract

Let F be a field of odd characteristic, E be a finite extension of F equipped an involution with subfield of fixed points E_0 containing F and V be a finite dimensional E-vector space with a non-degenerate hermitian form h. We show a link between the spinor norm in the unitary group $\mathrm{U}(V, h)$ and the calculus of determinants and discriminants. Then we show a formula which links the spinor norm in $\mathrm{U}(V, h)$ and the spinor norm in the orthogonal group $\mathrm{O}(V, b_h)$ defined by a non-degenerate symmetric bilinear form b_h associated to h.

1 Introduction

Let F be a field of characteristic not equal to 2 and let E be a finite extension of F, equipped with a non trial involution $^{-}$ which fixes all the elements of F. Denote by E_0 the subfield of fixed points of E by the involution. We fix a non-zero F-linear form μ_0 from E_0 to F and put $\mu = \mu_0 \circ \mathrm{tr}_{E/E_0}$, where tr_{E/E_0} is the trace form from E to E_0.

Let V be a finite dimensional vector space over E and let $h : V \times V \to E$ be a non-degenerate hermitian form on V. Considering V as an F-vector space, we have an associated non-degenerate symmetric bilinear form b_h defined by

$$b_h(x, y) = \mu(h(x, y)), \text{ for all } (x, y) \in V \times V.$$

Key words: Spinor norm, Unitary group, spin group, supercuspidal representation.
2010 AMS Mathematics classification: 11E57

Denote by $\mathrm{U}(V,h)$ the unitary group defined by the hermitian form h and by $\mathrm{O}(V,b_h)$ the orthogonal group defined by the symmetric form b_h. Then every element $\sigma \in \mathrm{U}(V,h)$ is clearly also an element of $\mathrm{O}(V,b_h)$.

Let sn be the spinor norm on the group $\mathrm{O}(V,b_h)$. It is a homomorphism from $\mathrm{O}(V,b_h)$ to $F^\times/(F^\times)^2$. The spinor norm has important applications [3]. Zassenhaus [6] found a direct definition of sn which links this norm with the calculus of determinants and discriminants: for each $\sigma \in \mathrm{O}(V,b_h)$, we have

$$\mathrm{sn}(\sigma) = \begin{cases} \mathrm{disc}(b_h) \text{ if } \sigma = -1, \\ \mathrm{disc}\left(b_h|_{\bigcup\limits_{i\geq 1} \ker(1+\sigma)^i}\right) \det_F\left(\frac{1+\sigma}{2}|_{\bigcap\limits_{i\geq 1} \mathrm{im}(1+\sigma)^i}\right) \text{ otherwise,} \end{cases} \tag{1.1}$$

where $\mathrm{disc}(b_h)$ and $\mathrm{disc}\left(b_h|_{\bigcup\limits_{i\geq 1} \ker(1+\sigma)^i}\right)$ are respectively the discriminant of b_h and the discriminant of the restriction of b_h to the subspace $\bigcup\limits_{i\geq 1} \ker(1+\sigma)^i$.

It is well known that there is an anti-hermitian form h' on V such that the unitary groups $\mathrm{U}(V,h')$ and $\mathrm{U}(V,h)$ coincide [1]. We summarize here Wall's construction [5] of a spinor norm on the unitary group $\mathrm{U}(V,h')$. Let σ be a non-trivial element of $\mathrm{U}(V,h)$. Denote by V_σ the image of the transformation $1-\sigma$. If V_σ is an E-vector subspace of dimension r then we say that σ is an *element of dimension* r. By definition, each element of dimension 1, denoted $s_{(v,\varphi)}$, is defined by

$$s_{(v,\varphi)}(x) = x - \varphi h'(v,x)v, \text{ for all } x \in V,$$

where v is a non-zero element of the space $V_{s_{(v,\varphi)}}$ and φ is an element of $E^\times$ such that $\varphi^{-1} - \bar{\varphi}^{-1} = h'(v,v)$. For each element $\sigma \in \mathrm{U}(V,h')$ of dimension $r > 0$, let f_σ be the sesquilinear form (with respect to the involution$^-$) on V_σ defined by

$$\begin{gathered} f_\sigma : V_\sigma \times V_\sigma \to E, \\ (x-\sigma(x), y-\sigma(y)) \mapsto h'(x-\sigma(x), y). \end{gathered}$$

Then σ can be written as a product of one-dimensional elements [5, Lemma 3],

$$\sigma = s_{(v_1,\varphi_1)} s_{(v_2,\varphi_2)} \ldots s_{(v_r,\varphi_r)},$$

where the vectors $v_1, v_2, ..., v_r$ form a orthogonal basis of V_σ with respect to the form f_σ and v_1 can be chosen as any non-isotropic vector of V_σ. Such a decomposition of σ is called *Cayley decomposition* of σ.

Let a be a fixed vector of V and suppose $\sigma \in \mathrm{U}(V,h')$ is an element of dimension $r > 0$ with a Cayley decomposition

$$\sigma = s_{(v_1,\varphi_1)} s_{(v_2,\varphi_2)} \ldots s_{(v_r,\varphi_r)},$$

where the vector $v_i, i = 1, 2, ..., r$, is chosen such that $h'(a, v_i)$ is either 0 or 1. Then the class $\varphi_1\varphi_2...\varphi_r E_0^\times$ in $E^\times/E_0^\times$ depends only on σ [5, Lemma 5]. This class is called *spinor norm of* σ, denoted by $\mathrm{sn}_E(\sigma)$:

$$\mathrm{sn}_E(\sigma) = \varphi_1 \ldots \varphi_r E_0^\times. \tag{1.2}$$

The spinor norm of the identity of $\mathrm{U}(V, h')$ is defined to be $E_0^\times$. Then we have a homomorphism sn_E of $\mathrm{U}(V, h')$ in $E^\times/E_0^\times$ [5, Lemma 6, Lemma 7], called *spinor norm* on $\mathrm{U}(V, h')$.

The goal of this work is to compare the spinor norm in the orthogonal group $\O(V, b_h)$ and the spinor norm in the unitary group $\mathrm{U}(V, h)$. A natural way to do so is to get a formula similar to Zassenhaus' formula (1.1).

Proposition. *Let $s = s_{(v,\varphi)}$ be an one-dimensional element of* $\mathrm{U}(V, h')$. *Then we have*

$$\operatorname{disc}\left(h'|_{\bigcup\limits_{n\geq 1} \ker(1+s)^n}\right) \det{}_E\left(\frac{1+s}{2}|_{\bigcap\limits_{n\geq 1} \mathrm{im}(1+s)^n}\right) = \varphi \mod (E_0^\times). \tag{1.3}$$

Theorem. *For all $\sigma \in \mathrm{U}(V, h')$, we have*

$$\mathrm{sn}(\sigma) = \mathrm{Norm}_{E/F}(\mathrm{sn}_E(\sigma)), \tag{1.4}$$

where $\mathrm{Norm}_{E/F}$ *is the homomorphism of $E^\times/E_0^\times$ in $F^\times/(F^\times)^2$ induced by the norm of E over F.*

Note that, for the case where E is a quadratic extension of F, this observation has been given in [4, Chapter 10, Theorem 1.5] with an incorrect proof. We give here another proof for this link in general case.

This paper is based on the research which is part of the doctoral dissertation [2] of the author. The results are useful in the study of supercuspidal representations of spin groups over a p-adic field, where some calculations arise involving the restriction of the spinor norm to unitary groups contained in the orthogonal group under study. The author is grateful to Corinne Blondel for her support, advice and interest in this work at various times.

2 Proof of the proposition

In order to prove the formula (1.3), we distinguish two cases: v is an isotropic vector and v is not one with respect to the form h'.

In the first case, the element $s = s_{(v,\varphi)}$ is called a *transvection* of V. We have $\varphi^{-1} - \bar{\varphi}^{-1} = h'(v, v) = 0$, hence φ belongs to $E_0^\times$ and the spinor norm sn_E is trivial at s. Now we calculate the left side of (1.3). Let x be an element of $\ker(1 + s)$. Then x belongs to the one-dimensional E-vector subspace of

V generated by v, *i.e.*, $x = kv$ for some $k \in E$. Since v is isotropic, we have $x+x = 0$ and then $x = 0$. It follows that the subspace $\bigcup_{n\geq 1} \ker\,(1+s)^n$ is trivial. Furthermore, there exists a basis $\{v, v', w_1, ..., w_{n-2}\}$ of V such that v' is also isotropic with respect to h', $h'(v,v') = 1$ and $h'(v, w_i) = 0, i = 1, ..., n-2$. Calculating the determinant in this basis, we have

$$\det_E\left(\frac{1+s}{2}\right) = \det_E \begin{pmatrix} 1 & -\frac{\varphi}{2} \\ 0 & 1 \end{pmatrix} = 1.$$

So we see that the formula (1.3) holds for the first case.

In the second case, we have an orthogonal decomposition $V = (Ev) \perp (Ev)^{\perp}$ where $(Ev)^{\perp}$ is the orthogonal complement of the line Ev in V with respect to h'. Let x be an element of $\ker(1+s)$. Then $x = kv$ for some $k \in E$ and we have $2kv - \varphi h'(v, kv)v = 0$. Since $\varphi^{-1} - \bar{\varphi}^{-1} = h'(v,v)$, we have

$$k[1 + \varphi\bar{\varphi}^{-1}]v = 0.$$

It follows that the subspace $\bigcup_{n\geq 1} \ker(1+s)^n$ is either zero or the line Ev. In the first situation, we have

$$\det_E\left(\frac{1+s}{2}\right) = \frac{1+\varphi\bar{\varphi}^{-1}}{2} \equiv \varphi \mod E_0^{\times}$$

while, in the second, we have $\varphi\bar{\varphi}^{-1} = -1$ and

$$\mathrm{disc}\,(h'|_{Ev}) = \left(\varphi^{-1} - \bar{\varphi}^{-1}\right) \equiv \varphi \mod E_0^{\times}.$$

Then it is easy to see that the formula (1.3) holds in these two situations.

3 Proof of the theorem

3.1 One-dimensional elements

Firstly we note that in the case where $\varphi\bar{\varphi}^{-1} = -1$ the element $s = s_{(v,\varphi)}$ is the reflection of V defined by the vector v with respect to h', *i.e.*, it is the linear transformation of V such that $s(v) = -v$ and $s(x) = x$ for all $x \in V$ such that $h'(v,x) = 0$. Otherwise the subspace $\bigcup_{n\geq 1} \ker(1+s)^n$ is zero as seen in the proof of the Proposition. Then by (1.1) we have

$$\mathrm{sn}(s) = \det_F\left(\frac{1+s}{2}\right) \mod (F^{\times})^2$$

and by (1.3) we have

$$\mathrm{sn}_E(s) = \det_E\left(\frac{1+s}{2}\right) \mod (E_0^{\times}).$$

This gives us the following lemma which is the affirmation of the formula (1.4) for one-dimensional elements which are not reflections.

Lemma 3.1. *Let* $s = s_{(v,\varphi)}$ *be an one-dimensional element of* $\mathrm{U}(V, h')$ *which is not a reflection. Then*

$$\mathrm{sn}(s) = \mathrm{Norm}_{E/F}\left(\mathrm{sn}_E(s)\right).$$

3.2 Quadratic case

We consider now the case where $E_0 = F$. In this case, sn and $\mathrm{Norm}_{E/F} \circ \mathrm{sn}_E$ are two homomorphisms of $\mathrm{U}(V, h')$ in $F^\times/(F^\times)^2$. Then it suffices to verify the equality (1.4) for the one-dimensional elements since they generate $\mathrm{U}(V, h')$.

Given Lemma 3.1, we only need to verify the formula for the reflections of V. Let $u = s_{(v,\varphi)}$ be a reflection of V, *i.e.*, $\varphi\bar{\varphi}^{-1} = -1$. Then the identity (1.4) becomes

$$\mathrm{disc}(b_h|_{Ev}) = \mathrm{Norm}_{E/F}(\mathrm{disc}(h'|_{Ev})) \mod (F^\times)^2.$$

Note that, in this case, we have $h' = \delta h$ where $\delta \in E$ and $\{1, \delta\}$ forms a orthogonal F-basis of E with respect to b_h. Identifying the space Ev with E, the restriction of h to this space is a hermitian form on E. Then we have $h(x, y) = ax\bar{y}, \forall x, y \in E$, for some $a \in E_0$. It follows that $\mathrm{disc}(h'|_E) = a\delta \mod (E_0^\times)$ and

$$\mathrm{disc}(b_h|_E) = \det{}_F \begin{pmatrix} 2a & 0 \\ 0 & -2a\delta^2 \end{pmatrix} \mod (F^\times)^2 = \mathrm{Norm}_{E/F}(a\delta) \mod (F^\times)^2.$$

That means the formula (1.4) holds for the quadratic case:

Proposition 3.2. *If* $E_0 = F$ *then we have*

$$\mathrm{sn}(u) = \mathrm{Norm}_{E/F}(\mathrm{sn}_E(u)), \textit{ for all } u \in \mathrm{U}(V, h').$$

Remark 3.3. Let u be an element of $\mathrm{U}(V, h')$. Suppose $\det_E(u) = \alpha$. Then $\mathrm{Norm}_{E/E_0}(\alpha) = 1$. By Hilbert's Theorem 90, there exists a unique element $\beta \in E^\times$ up to a scalar in E_0 such that $\alpha = \beta\bar{\beta}^{-1}$. Then we have a homomorphism

$$\mathrm{Hil} : \mathrm{Norm}_{E/E_0}^{-1}(1) \to E^\times/E_0^\times, \alpha \mapsto \beta E_0^\times, \text{ where } \alpha = \beta\bar{\beta}^{-1}.$$

The spinor norm sn_E is in fact the composition of the determinant and the homomorphism Hil, *i.e.*, we have

$$\mathrm{sn}_E(u) = \mathrm{Hil}(\det{}_E(u)), \forall u \in U(V, h').$$

In order to prove this identity we only need to verify it for the one-dimensional elements of $\mathrm{U}(V, h')$. Let $u = s_{(v,\varphi)}$ be an one-dimensional element of $\mathrm{U}(V, h')$.

If v is an isotropic vector then the identity is evident since $\det_E(u) = 1$ and the spinor norm of u is trivial by definition. If v is non-isotropic then we have

$$\det_E(u) = 1 - \varphi h'(v, v) = 1 - \varphi(\varphi^{-1} - \bar{\varphi}^{-1}) = \varphi\bar{\varphi}^{-1}.$$

It follows that $\mathrm{Hil}(\det_E(u)) = \varphi E_0^\times = \mathrm{sn}_E(u)$. With this point of view on the spinor norm on $\mathrm{U}(V, h')$, we can see that Proposition 3.2 is similar to [4, Chapter 10, Theorem 1.5]. However the proof in *loc.cit.* is not correct since $\tilde{\sigma} \neq \alpha_\beta$ in its notations.

3.3 General case

We prove now the Theorem in the general case. For all $x, y \in V$, put

$$h_0(x, y) = \mathrm{tr}_{E/E_0}(h(x, y)).$$

Then h_0 is a non-degenerate symmetric bilinear form on the E_0-vector space V. Denote $\mathrm{SO}(V, h_0)$ the group of the rotations of V with respect to h_0 and sn_{E_0} the spinor norm in $\mathrm{SO}(V, h_0)$. Note that we have

$$\mathrm{U}(V, h) \subset \mathrm{SO}(V, h_0) \subset \mathrm{O}(V, b_h)$$

and, by Proposition 3.2, we have

$$\mathrm{sn}_{E_0}(u) = \mathrm{Norm}_{E/E_0}(\mathrm{sn}_E(u)), \forall u \in \mathrm{U}(V, h).$$

For the passage from E_0 to F, we use the transfer properties of the Witt ring of quadratic spaces [4, Chapter 9, §5]: Consider E_0 as an E_0-vector space and denote ϕ_0 the symmetric bilinear form on E_0 defined by

$$\phi_0(x, y) = xy, \forall x, y \in E_0.$$

Then $\mu_0 \circ \phi_0$ is a symmetric bilinear form on F-vector space E_0. Put $\zeta = \mathrm{disc}(\mu_0 \circ \phi_0)$. Let ϕ be a symmetric bilinear form on an E_0-vector space W of dimension n. Then $\mu_0 \circ \phi$ is also a symmetric bilinear form on F-vector space W. In this situation, we have [4, Chapter 9, Theorem 5.12]

$$\mathrm{disc}(\mu_0 \circ \phi) = (\zeta)^n \mathrm{Norm}_{E_0/F}(\mathrm{disc}(\phi)).$$

Return to our situation, let s be a reflection of the E_0-vector space V with respect to h_0. Using Zassenhaus' formula (1.1) and the transfer property above we have

$$\mathrm{sn}(s) = \zeta \mathrm{Norm}_{E_0/F}(\mathrm{sn}_{E_0}(s)).$$

Since the reflections generate the group $\mathrm{SO}(V, h_0)$ [1], we obtain

$$\mathrm{sn}(u) = \mathrm{Norm}_{E_0/F}(\mathrm{sn}_{E_0}(u)), \text{ for all } u \in \mathrm{SO}(V, h_0).$$

This completes the proof of the Theorem.

References

[1] J. Dieudonné, La géométrie des groupes classiques, Springer-Verlag, Berlin, 1955.

[2] N.V. Dinh, Construction de représentations supercuspidales des groupes spinoriels définis sur des corps p-adiques au moyen de types semi-simples, PhD Thesis, Université Paris Diderot, 2013.

[3] M. Kneser, *Orthogonale Gruppen über algebraischen Zahlkörpern*, J. Reine Angew. Math., 1956, Vol. 196, pp. 213 – 220.

[4] W. Scharlau, Quadratic and Hermitian Forms, Grundlehren der Mathematischen Wissenschaften 270, Springer-Verlag, Berlin, 1985.

[5] G.E. Wall, *The structure of a unitary factor group*, Publ. Math. Inst. Hautes Études Sci., 1959, Vol. 1, pp. 7 – 23.

[6] H. Zassenhaus, *On the Spinor norm*, Archiv der Mathematik, 1962, Vol. 13, pp. 434 – 451.