ON REVERSIBILITY OF RINGS WITH INVOLUTION

Usama A. Aburawash and Muna E. Abdulhafed

Department of Mathematics and Computer Science
Faculty of Science, Alexandria University, Alexandria, Egypt
e-mail: aburawash@alexu.edu.eg

Faculty of Arts and Science, Azzaytuna University, Tarhunah, Libya
muna.am2016@gmail.com

Abstract

Let $R$ be a ring with involution $\ast$. We give the notion of central $\ast$-reversible $\ast$-rings which generalizes weakly $\ast$-reversible $\ast$-rings. Moreover, we introduce the class of weakly $\ast$-rings which is a generalization of central $\ast$-reversible $\ast$-rings and investigate their properties. Further, a generalization of the class of quasi-$\ast$-IFP $\ast$-rings is given; namely weakly quasi-$\ast$-IFP $\ast$-rings. Since every $\ast$-reversible $\ast$-ring is central $\ast$-reversible, we give sufficient conditions for central $\ast$-reversible, weakly $\ast$-reversible and weakly quasi-$\ast$-IFP $\ast$-rings to be $\ast$-reversible and some examples are given to illustrate these situations. Finally, we show that the properties of $\ast$-reversible, central $\ast$-reversible, weakly $\ast$-reversible and weakly quasi-$\ast$-IFP can be transfer to some extensions of the $\ast$-ring.

1 Introduction

Throughout this paper, a ring will always mean an associative ring with unity unless otherwise stated. A ring $R$ is said to be $\ast$-ring if on $R$ there is defined an involution $\ast$; that is an anti-isomorphism of order two. The right annihilator of the nonempty set $A$ of $R$ is denoted by $r_R(A)$ and the right $\ast$-annihilator of $A$ is denoted by $r_{R}(A) = \{x \in R \mid Ax = Ax^* = 0\}$. If there is no ambiguity, we omit the subsuffix $R$. A $\ast$-ideal (self-adjoint) $I$ of $R$ is an ideal closed under involution. A self adjoint idempotent; $e^2 = e = e^*$, is

Key words: Involution; $\ast$-Reversible; Central $\ast$-reversible; Weakly $\ast$-reversible; Quasi-$\ast$-IFP; Weakly $\ast$-IFP; Weakly quasi-$\ast$-IFP $\ast$-rings.

2010 AMS Mathematics classification:
called projection. A nonzero element \( a \) of a *-ring \( R \) is called *-zero divisor if \( ab = 0 = a^*b \), for some nonzero element \( b \in R \) and \( R \) is *-domain if it has no nonzero *-zero divisors, from [6]. A *-ring \( R \) is said to be Abelian (*-Abelian) if every idempotent (projection) of \( R \) is center. A *-ring \( R \) is reduced if it has no nonzero nilpotent elements. A ring \( R \) is called semicommutative or has (IFP) if for all \( a, b \in R \), \( ab = 0 \) implies \( aRb = 0 \) (equivalently \( r(a) \) is an ideal of \( R \) for all \( a \in R \) (see [10]). A *-ring \( R \) is said to have *-IFP if for all \( a, b \in R \), \( ab = 0 \) implies \( aRb^* = 0 \) (equivalently \( r(a) \) is a *-ideal of \( R \) for all \( a \in R \) (see [4]). From [13], recall a ring \( R \) is weakly semicommutative if for all \( a, b \in R \), \( ab = 0 \) implies \( ab \) is a nilpotent element for each \( r \in R \). By [7], a ring \( R \) is called reversible if for all \( a, b \in R \), \( ab = 0 \) implies \( ba = 0 \). According to [3], a *-ring \( R \) is called *-reversible if for all \( a, b \in R \), \( ab = 0 = ab^* \) implies \( ba = 0 \), and \( R \) has quasi-*-IFP if for all \( a, b \in R \), \( ab = ab^* \) implies \( aRb = 0 \). From [5], an element \( a \) of a *-ring \( R \) is called *-nilpotent if \( a^n = (aa^*)^n = 0 \), for some positive integers \( m \) and \( n \). \( R \) is *-reduced if it has no nonzero *-nilpotent elements. Following [9], a *-ring \( R \) is called Baer *-ring if the right annihilator of every nonempty subset of \( R \) is generated, as a right ideal, by a projection. By [5], a *-ring \( R \) is called *-Baer *-ring if the *-right annihilator of every nonempty subset of \( R \) is generated, as a biideal, by a projection. From [8] a ring \( R \) is central reversible rings if for all \( a, b \in R \), \( ab = 0 \) implies \( ba \) belongs to the center of \( R \) and a ring \( R \) is called weakly reversible if \( ab = 0 \) implies \( Rbra \) is nil left ideal of \( R \), for all \( a, b, r \in R \), from [11]. The natural numbers and the integers will be denoted by \( \mathbb{N} \) and \( \mathbb{Z} \), respectively. \( M_n(R) \) will denote the full matrix ring of all \( n \times n \) matrices over the ring \( R \), while \( T_n(R) \) \((T_n(E(R)) \) will denote the \( n \times n \) upper triangular matrix ring (with equal diagonal elements) over \( R \).

In this paper, we introduce central and weakly *-reversible *-rings, both are proper generalizations of *-reversible *-rings. Moreover, the class of weakly *-reversible *-rings contains strictly central *-reversible *-rings. We also prove that central *-reversible *-rings are *-Abelian and there exists a *-Abelian *-ring which is not central *-reversible. Clearly *-reversible *-rings are quasi-*-IFP and example is given to show that the converse is not true and another example shows that commutative weakly *-reversible *-rings do not necessarily have quasi-*-IFP. It is also shown that if \( R \) is a commutative *-ring, then \( T_{nE}(R) \) is weakly *-reversible (weakly quasi-*-IFP) *-ring. Moreover, weakly quasi-*-IFP condition is given for *-rings which generalizes quasi-*-IFP. We show also that commutative weakly quasi-*-IFP *-rings may not be quasi-*-IFP. Moreover, for a *-Armendariz *-ring \( R \), we prove that \( R \) is *-reversible (central *-reversible) if and only if the polynomial *-rings \( R[x] \) is *-reversible (central *-reversible) if and only if the Laurent polynomial *-ring \( R[x; x^{-1}] \) is *-reversible (central *-reversible). Furthermore, it is proved that \( R \) is *-reversible (central *-reversible) if and only if the Dorroh extension \( D(R, \mathbb{Z}) \) of
2 Central *-Reversible *-Rings

In this section, we introduce and study the class of central *-reversible *-rings, which is a generalization of *-reversible *-rings. We start by giving the main definition.

Definition. A *-ring \( R \) is called central *-reversible if for all \( a, b \in R \), \( ab = 0 = ab^* \) implies \( ba \) is central in \( R \). Consequently, \( b^*a \) is central in \( R \).

Clearly, a central reversible *-ring is central *-reversible and a *-reversible *-ring is central *-reversible. However, the next result shows that \( T_3(E)(R) \), in general, is central *-reversible but not *-reversible.

Proposition 1. Let \( R \) be a commutative *-ring, then the ring

\[
T_3(E)(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}
\]

with involution defined as \( \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \) is central *-reversible *-ring.

Proof. Let \( x = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \) and \( y = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in T_3(E)(R) \). If \( xy = 0 = xy^* \), then we have the following equations:

\[
\begin{align*}
a_1a_2 &= 0 \\
a_1b_2 + b_1a_2 &= 0, \quad a_1d_2 + b_1a_2 &= 0 \\
a_1c_2 + b_1d_2 + c_1a_2 &= 0, \quad a_1c_2 + b_1b_2 + c_1a_2 &= 0 \\
a_1d_2 + d_1a_2 &= 0, \quad a_1b_2 + d_1a_2 &= 0.
\end{align*}
\]

Hence \( yx = \begin{pmatrix} 0 & 0 & b_2d_1 - b_1d_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0 \), is central and consequently \( T_3(E)(R) \) is central *-reversible. On the other hand, \( T_3(E)(R) \) is not *-reversible, since \( yx \neq 0 \), while the converse is clear from [3, Example 3.8].

In general, Proposition 1 is not true for \( n \geq 4 \) which is clear from the following example.
Example 1. Consider the *-ring $T_{4E}(\mathbb{Z})$ with the involution * defined as:

$$
\begin{pmatrix}
0 & a & a_{13} & a_{14} \\
0 & a & a_{23} & a_{24} \\
0 & 0 & a & a_{34} \\
0 & 0 & 0 & a
\end{pmatrix}^* =
\begin{pmatrix}
a & a_{34} & a_{24} & a_{14} \\
0 & a & a_{23} & a_{13} \\
0 & 0 & 0 & a \\
0 & 0 & 0 & a
\end{pmatrix}.
$$

The matrices $A = \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$ and $B = \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$ satisfies $AB = 0 = AB^*$, but $BA = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$ is not central and so $T_{4E}(\mathbb{Z})$ is not central*-reversible.

It is clear that each central reversible is central *-reversible. However, the converse is true when the ring has *-IFP as shown in the next result.

Proposition 2. Let $R$ be a *-ring. If $R$ is central *-reversible and has *-IFP, then $R$ is central reversible.

Proof. Obvious, since $ab = 0$, implies $aRb^* = 0$, by *-IFP property, and $R$ is central reversible. □

Recall that a *-ring $R$ is *-semiprime if and only if it is semiprime (see ([1])). Next, we give some particular conditions for a central *-reversible *-ring to be *-reversible.

Proposition 3. A semiprime central *-reversible *-ring is *-reversible.

Proof. Assume that $R$ is a semiprime central *-reversible *-ring. If $ab = ab^* = 0$, then $ba$ is central and consequently $baRba = 0$. Form semiprimeness, we get $ba = 0$ and so $R$ is *-reversible. □

Proposition 4. If $R$ is a *-Baer and central *-reversible *-ring, then $R$ is *-reversible.

Proof. Let $R$ be a *-Baer *-ring and $ab = 0 = ab^*$, then there exists a projection $e \in R$ such that $r_*(a) = eRe$. We have $ae = 0$ and $b = ebe = eb$, since $b \in r_*(a) = eRe$. Hence $ba = eba = bae = 0$, since $ba$ is central, and so $R$ is *-reversible. □

Since each Bear *-ring is *-Bear, we have the following corollary.

Corollary 1. If $R$ is a Baer and central *-reversible *-ring, then $R$ is *-reversible.
Furthermore, the class of central \(*\)-reversible \(*\)-rings is clearly closed under direct sums (with changeless involution) and under taking \(*\)-subrings by [3], since every \(*\)-reversible \(*\)-ring is central \(*\)-reversible.

**Proposition 5.** The class of central \(*\)-reversible \(*\)-ring is closed under direct sums and under taking \(*\)-subrings.

**Proposition 6.** Let \(R\) be a \(*\)-ring and \(e\) be a central projection of \(R\). Then \(eR\) and \((1-e)R\) are \(*\)-reversible if and only if \(R\) is \(*\)-reversible.

**Proof.** It suffices to show the necessity by [3, Proposition 3.15]. Let \(ab = ab^* = 0\) with \(a, b \in R\), then \(cab = cab^* = 0\) and \((1-e)ab = (1-e)ab^* = 0\). By assumption, we have \(bea = 0\) and \(b(1-e)a = 0\). Hence \(ba = bea + |b(1-e)a| = 0\) and so \(R\) is \(*\)-reversible. □

By a similar proof as **Proposition 6**, and using **Proposition 5**, the following corollary is immediate.

**Corollary 2.** Let \(R\) be a \(*\)-ring and \(e\) be a central projection of \(R\). Then \(eR\) and \((1-e)R\) are central \(*\)-reversible if and only if \(R\) is central \(*\)-reversible.

Recall that a \(*\)-ideal \(I\) of a \(*\)-ring \(R\) is \(*\)-nil if each element of \(I\) is \(*\)-nilpotent.

Obviously, each \(*\)-nil ideal is nil. The following example shows that the converse is not always true.

**Example 2.** For the \(*\)-ring \(R = \mathbb{M}_2(\mathbb{Z})\) of all \(2 \times 2\) matrices over the integers \(\mathbb{Z}\) with transpose of matrices as involution, the nonzero elements of the form
\[
\begin{pmatrix}
0 & x \\
0 & 0
\end{pmatrix}
\]
are all nilpotent but not \(*\)-nilpotent, since \(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}^2 = 0\) but \(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} = \begin{pmatrix} x^2 & 0 \\ 0 & 0 \end{pmatrix} \neq 0\).

We note that the homomorphic image of a central \(*\)-reversible \(*\)-ring need not be central \(*\)-reversible as seen from the following example.

**Example 3.** Let \(D\) be a \(*\)-division ring, \(R = D[x,y]\) and \(I = \langle xy \rangle\), where \(xy \neq yx\). Since \(R\) is \(*\)-domain, \(R\) is central \(*\)-reversible. On the other hand, \((x+I)(y+I)\) and \((x+I)^*(y+I) = (x+I)(y+I)\) are both zero. But \((y+I)(x+I)\) is not central in \(R/I\), hence \(R/I\) is not central \(*\)-reversible.

Moreover, the next example shows that if the homomorphic image of a \(*\)-ring \(R\) is central \(*\)-reversible, then \(R\) need not be central \(*\)-reversible.
Example 4. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $F$ is a field, with the adjoint involution $^*$. Consider the $*$-ideal $I = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ of $R$. Then $R/I$ is central $*$-reversible, because of the commutativity property of $R/I$. For $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$ where $B^* = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \in R$, we have $AB = 0 = AB^*$. Consider $C = \begin{pmatrix} c_1 & c_2 \\ 0 & c_3 \end{pmatrix} \in R$ with $c_1 \neq c_3$. It is clear that $CBA \neq BAC$ and therefore $R$ is not central $*$-reversible.

Our next endeavour is to give a condition on the homomorphic image of a $*$-ring to be central $*$-reversible. Recall that a $*$-ring $R$ is called unit-central, if all unit elements of $R$ are central in $R$. Moreover, we show that every unit central $*$-ring is $*$-Abelian.

Proposition 7. Let $R$ be a unit-central $*$-ring. If $I$ is a $*$-nil ideal of $R$, then $R/I$ is central $*$-reversible.

Proof. Let $a, b \in R$ with $(a + I)(b + I) = (a + I)(b + I)^* = I$. Then $ab \in I, ab^* \in I$ and so there exists a positive integers $m, n, p$ and $q$ such that $(ab)^m = 0$, $((ab)(ab)^*)^n = 0$, $(ab)^p = 0$ and $((ab^*)(ab^*)^*)^q = 0$. It follows that $(ba)^{m+1} = 0$, whence $1 - ba$ is unit and so central by hypothesis. Thus $rba = bar$ for any $r \in R$ and therefore $(b + I)(a + I)$ is central in $R/I$. 

Since each $*$-reversible $*$-ring is central $*$-reversible and each $*$-domain is $*$-reversible, by [3, Example 3.2], we have immediately the following corollary.

Corollary 3. Every $*$-domain is a central $*$-reversible $*$-ring.

The converse of Corollary 3 is not true by Example 4. However, the converse is true for $*$-prime $*$-rings as follows.

Proposition 8. Let $R$ be a $*$-ring. Then $R$ is $*$-prime and central $*$-reversible if and only if it is $*$-domain.

Proof. Let $R$ be $*$-prime and central $*$-reversible and $ab = ab^* = 0$ for some $a, b \in R$. We have $rab = rab^* = 0$ for every $r \in R$ and so bra and $b^*ra$ are central. Since $bratb = 0$ and $bratb^* = 0$ for all $t \in R$, then $a = 0$ or $b = 0$ and $R$ is a $*$-domain. The converse is obvious by Corollary 3.

It is well known from [3, Corollary 3.7] that every $*$-reversible $*$-ring is $*$-Abelian. Similarly, we have the same result for central $*$-reversible case.
Proposition 9. A central *-reversible *-ring $R$ is *-Abelian.

Proof. Let $e^2 = e = e^* \in R$. for any $r \in R$, $(re - ere)(1 - e) = (re - ere)(1 - e)^* = 0$ implies $(1 - e)(re - ere) = re - ere$ is central. Commuting $re - ere$ by $e$ we get $re - ere = 0$. Similarly for any $r \in R$, $(r^*e - er^*e)(1 - e) = (r^*e - er^*e)(1 - e)^* = 0$ implies $r^*e - er^*e = 0$. Therefore $re = ere = er$ and $R$ is *-Abelian. □

The next example shows that the reverse implication of Proposition 9 is not true in general; that is there exists a *-Abelian *-ring which is not central *-reversible, and hence is not *-reversible.

Example 5. The only projections of the *-ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a \equiv c (\text{mod} \ 2), b \equiv 0 (\text{mod} \ 2), a, b, c \in \mathbb{Z} \right\}$ under adjoint involution * are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and so $R$ is *-Abelian. On the other hand, for $x = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in R$ with $xy = xy^* = 0$, we have $yx = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$ is not central and so $R$ is not central *-reversible.

3 Weakly *-Reversible *-Rings

In this section, we introduce another generalization for *-reversible; namely weakly *-reversible *-rings.

Definition. A *-ring $R$ is called weakly *-reversible if for all $a, b, r \in R$, $ab = ab^* = 0$, implies $Rbr$ is a nil left (equivalently, $braR$ is a nil right) ideal of $R$. Consequently, $Rb^*ra$ is a nil left (equivalently, $b^*raR$ is a nil right) ideal of $R$.

Each commutative *-ring is weakly reversible. Clearly, each weakly reversible *-ring is weakly *-reversible. The converse is true when the ring has *-IFP as shown in the following.

Proposition 10. Let $R$ be a *-ring. If $R$ is weakly *-reversible and has *-IFP, then $R$ is weakly reversible.

Proof. Obvious, since $ab = 0$, implies $aRb^* = 0$, by the *-IFP property, and $R$ is weakly reversible. □

Moreover, we can easily prove the following result.

Proposition 11. The class of weakly *-reversible *-ring is closed under direct sums (with changeless involution) and under taking *-subrings.
Proposition 12. For a commutative *-ring $R$, $T_{nE}(R)$ is a weakly *-reversible *-ring, with involution * defined by fixing the two diagonals considering the diagonal right / left lower as symmetric ones and interchange the symmetric elements about it; that is

\[
\begin{pmatrix}
  a & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\
 0 & a & a_{23} & \cdots & a_{2(n-1)} & a_{2n} \\
 0 & 0 & a & \cdots & \cdots & a_{3n} \\
 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & a_{(n-1)n} & a
\end{pmatrix}^* = 
\begin{pmatrix}
  a & a_{(n-1)n} & a_{(n-2)n} & \cdots & a_{2n} & a_{1n} \\
 0 & a & a_{23} & \cdots & a_{2(n-1)} & a_{1(n-1)} \\
 0 & 0 & a & \cdots & \cdots & a_{1(n-2)} \\
 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & a_{12} & a
\end{pmatrix}
\]

Proof. Let $R$ be weakly *-reversible and $A = \begin{pmatrix}
  a & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\
 0 & a & a_{23} & \cdots & a_{2(n-1)} & a_{2n} \\
 0 & 0 & a & \cdots & \cdots & a_{3n} \\
 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & a_{12} & a
\end{pmatrix}$, $B = \begin{pmatrix}
  b & b_{12} & b_{13} & \cdots & b_{1n} \\
 0 & b & b_{23} & \cdots & b_{2n} \\
 0 & 0 & b & \cdots & \cdots & b_{3n} \\
 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & b
\end{pmatrix} \in T_{nE}(R)$ satisfy $AB = 0 = AB^*$. Hence $ab = 0 = ab^*$ and since $R$ is weakly *-reversible, then for $C = \begin{pmatrix}
  c & c_{12} & c_{13} & \cdots & c_{1n} \\
 0 & c & c_{23} & \cdots & c_{2n} \\
 0 & 0 & c & \cdots & \cdots & c_{3n} \\
 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & c
\end{pmatrix}$ and $D = \begin{pmatrix}
  d & d_{12} & d_{13} & \cdots & d_{1n} \\
 0 & d & d_{23} & \cdots & d_{2n} \\
 0 & 0 & d & \cdots & \cdots & d_{3n} \\
 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & d
\end{pmatrix} \in T_{nE}(R)$, there exists $k \in \mathbb{N}$, with $(cbda)^k = 0$. Thus

\[
(CBDA)^k = \begin{pmatrix}
  0 & \star & \cdots & \star \\
 0 & 0 & \star & \cdots & \star \\
 0 & 0 & 0 & \cdots & \star \\
 0 & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
and $(CBDA)^{kn} = 0$ follows and $T_{nE}(R)$ is weakly *-reversible. [$\star$ denotes an element of $R$] \qed
Next, the given example shows that there exists a weakly *-reversible and quasi *-IFP *-ring which is not *-reversible.

**Example 6.** Let $R$ be a commutative *-ring. Then the *-ring

$$T_{3E}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\},$$

is weakly *-reversible by Proposition 12, for some $a \neq 0$. For $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have $AB = 0 = AB^*$ and $BA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so $T_{3E}(R)$ is not *-reversible, while it has quasi-*-IFP.

We note that if $R$ is a commutative then the *-ring

$$T_{nE}(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R, n \geq 3 \right\},$$

is not *-reversible by [3, Example 3.8] and is weakly *-reversible by Proposition 12. Moreover, it is clear that $T_{4E}(R)$ is not quasi-*-IFP and so $T_{nE}(R)$ is not quasi-*-IFP for $n \geq 4$.

The next example demonstrates that the condition $T_{nE}(R)$ in Proposition 12, cannot be weakened to the full matrix *-ring $M_n(R)$, where $n$ is any integer bigger than 1.

**Example 7.** Let $R$ be a weakly *-reversible *-ring and $n$ any integer bigger than 1, then $M_2(R)$, with adjoint involution, is not weakly *-reversible. For $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $AB = 0 = AB^*$ and for $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(R)$, we see that $RBCA = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$ is not nil.

The following result shows that the class of central *-reversible *-rings lies properly between the classes of *-reversible and weakly *-reversible *-rings.

**Theorem 1.** Let $R$ be a *-ring and consider the following conditions.

1. $R$ is *-reversible.
2. $R$ is central *-reversible.
3. \( R \) is weakly \( * \)-reversible.

Then (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3).

**Proof.**

(1) \( \Rightarrow \) (2): Clearly.

(2) \( \Rightarrow \) (3): Let \( a, b \in R \) with \( ab = ab^* = 0 \). Then for all \( s \in R \), \( sab = sab^* = 0 \) and \( bsa \) is central, since \( R \) is central \( * \)-reversible. Hence \( (rbsa)^2 = (rbsa)(rbsa) = r(bsa)r(bsa) = rr bs(ab)sa = 0 \), for all \( r, s \in R \) and \( R \) is weakly \( * \)-reversible.

\( \square \)

The converse of **Theorem 1** is not true by **Examples** 1 and 6. However, from **Corollary 3** and **Theorem 1** we get the following corollary.

**Corollary 4.** Every \( * \)-domain is a weakly \( * \)-reversible \( * \)-ring.

### 4 Weakly quasi-\( * \)-IFP

Here, weakly quasi-\( * \)-IFP \( * \)-rings are introduced as generalization for the class of quasi-\( * \)-IFP \( * \)-rings. First, we introduce weakly \( * \)-IFP \( * \)-rings.

**Definition.** A \( * \)-ring \( R \) is called weakly \( * \)-IFP if for all \( a, b \in R \), \( ab = 0 \) implies \( arb^* \in \text{nil}(R) \) for all \( r \in R \).

Each commutative \( * \)-ring is weakly \( * \)-IFP. As before, one can easily prove the following result.

**Proposition 13.** The class of weakly \( * \)-IFP \( * \)-ring is closed under direct sums (with changeless involution) and under taking \( * \)-subrings.

**Proposition 14.** For a commutative \( * \)-ring \( R \), \( T_{nE}(R) \) is weakly \( * \)-IFP, with involution \( * \) given in **Proposition 12**.

**Proof.** Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \in T_{nE}(R) \) with \( AB = 0 \), where \( 1 \leq i \leq j \leq n \), then we have \( ab = 0 \), where \( a \) and \( b \) are the diagonal elements of \( A \) and \( B \), respectively. Since \( R \) is weakly \( * \)-IFP, there exists \( k \in \mathbb{N} \) such that \( (acb)^k = 0 \) for all \( C = (c_{ij}) \in T_{nE}(R) \), where \( c \) is the diagonal element of \( C \). Hence \( ((ACB^*)^k)^n = 0 \) and \( T_{nE}(R) \) is weakly \( * \)-IFP.

\( \square \)

It is clear that every \( * \)-ring having \( * \)-IFP is weakly \( * \)-IFP while the converse is not always true as shown by the following example.
Example 8. The *-ring $T_{3E}(\mathbb{Z})$ with the involution * given by:

$$
\begin{pmatrix}
a & b & c \\
0 & a & d \\
0 & 0 & a 
\end{pmatrix}
\begin{pmatrix}
a d c \\
0 & a & b \\
0 & 0 & a 
\end{pmatrix}
= \begin{pmatrix}
a d c \\
0 & a & b \\
0 & 0 & a 
\end{pmatrix}
$$

is weakly *-IFP by Proposition 14. For $A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix}$ and $B = \begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix}$, we have $AB = 0$ and $ARB^* = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix} \neq 0$, so $T_{3E}(\mathbb{Z})$ has not *-IFP.

By the way, there exists a weakly IFP *-ring which is not weakly *-IFP as in the next example.

Example 9. Let $\mathbb{F}$ be a field and consider the *-ring $R = \mathbb{F} \oplus \mathbb{F}$, with the exchange involution $(a, b)^* = (b, a)$, for all $a, b \in \mathbb{F}$. $R$ is clearly weakly IFP and is not weakly *-IFP.

Next, we define weakly quasi-*-IFP *-rings

**Definition.** A *-ring $R$ is said to be weakly quasi-*-IFP if for all $a, b \in R$, $ab = 0 = ab^*$ implies $arb$ is a nilpotent element for each $r \in R$. Consequently $arb^*$ is also nilpotent.

Each commutative *-ring is weakly quasi *-IFP. Clearly, each weakly IFP *-ring is weakly quasi-*-IFP. The converse is true when the ring has *-IFP as shown in the following.

**Proposition 15.** Let $R$ be a *-ring. If $R$ is weakly quasi-*-IFP and has *-IFP, then $R$ is weakly IFP.

**Proof.** Clearly, since $ab = 0$, implies $aRb^* = 0$, by the *-IFP property, and $R$ is weakly quasi-*-IFP.

Moreover, the class of weakly quasi-*-IFP *-ring is closed under direct sums (using changeless involution) and under taking *-subrings.

**Proposition 16.** The class of weakly quasi-*-IFP *-ring is closed under direct sums and under taking *-subrings.

By a proof similar to Proposition 12, we get the following.

**Proposition 17.** If $R$ is a commutative *-ring, then $T_{nE}(R)$ is weakly quasi-*-IFP, with involution * given in Proposition 12.
Note that if $R$ is a commutative *-ring then the *-ring $T_nE(R) =$ \[
\begin{pmatrix}
     a & a_{12} & a_{13} & \cdots & a_{1n} \\
     0 & a & a_{23} & \cdots & a_{2n} \\
     0 & 0 & a & \cdots & a_{3n} \\
     \vdots & \vdots & \vdots & \ddots & \vdots \\
     0 & 0 & 0 & \cdots & a
\end{pmatrix}
\] with $a_{ij} \in R, n \geq 3$, is not *-reversible by [3, Example 3.8] and is weakly quasi-*-IFP by Proposition 17. However, it is clearly that $T_4E(R)$ is not quasi-*-IFP and so $T_nE(R)$ is not quasi-*-IFP for $n \geq 4$.

The next example demonstrates that the condition $T_nE(R)$ in Proposition 17, cannot be weakened to the full matrix *-ring $M_n(R)$, where $n > 1$.

**Example 10.** $\mathbb{Z}$ is weakly quasi-*-IFP *-ring with identical involution, while the *-ring $M_2(\mathbb{Z})$ with adjoint involution * is not weakly quasi-*-IFP. Indeed, the matrices $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ satisfy $AB = 0 = AB^*$ and for $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(R)$, we have $ACB = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ is not nilpotent.

It is well known that every *-reversible *-ring has quasi-*-IFP by [3, Proposition 3.6]. Next, we prove that central *-reversible *-rings are weakly quasi-*-IFP.

**Theorem 2.** Let $R$ be a *-ring and consider the following conditions.

1. $R$ is *-reversible.
2. $R$ is central *-reversible.
3. $R$ is weakly quasi-*-IFP.

Then (1) $\implies$ (2) $\implies$ (3).

**Proof.**

(1) $\implies$ (2). Is clear.

(2) $\implies$ (3). If $a, b \in R$ satisfy $ab = ab^* = 0$, then $ba$ is central and $(arb)^2 = 0$. Hence $arb$ is nilpotent for all $r \in R$ and $R$ is weakly quasi-*-IFP.

The converse of Theorem 2 is not true by Examples 1 and 6. Moreover, from Corollary 3 and Theorem 2 we have the following result.

**Corollary 5.** Every *-domain is a weakly quasi-*-IFP *-ring.
From Proposition 4 we have immediately the following corollary.

**Corollary 6.** If \( R \) is a \(*\)-Baer and central \(*\)-reversible \(*\)-ring, then \( R \) has quasi \(*\)-IFP.

From [8, Proposition 2.20], if \( R \) is central reduced (that is every nilpotent element is central), then \( T(R, R) \) is central reversible and from [3, Proposition 3.14], if \( R \) is \(*\)-reduced and \(*\)-reversible, then \( T(R, R) \), with componentwise involution, is \(*\)-reversible. Accordingly, we have the following corollaries.

**Corollary 7.** If the \(*\)-ring \( R \) is central reduced \(*\)-ring then \( T(R, R) \) is central \(*\)-reversible.

**Corollary 8.** If the \(*\)-ring \( R \) is reduced then \( T(R, R) \) is central \(*\)-reversible.

**Corollary 9.** If the \(*\)-ring \( R \) is \(*\)-reduced and \(*\)-reversible then \( T(R, R) \), with componentwise involution, is central \(*\)-reversible.

**Corollary 10.** If the \(*\)-ring \( R \) is reduced and \(*\)-reversible then \( T(R, R) \), with componentwise involution, is central \(*\)-reversible.

By [11, Corollary 2.4], \( R \) is weakly reversible if and only if its trivial extension \( T(R, R) \) is weakly reversible and from Proposition 12, we have the following corollaries.

**Corollary 11.** If \( R \) is weakly reversible then \( T(R, R) \) is weakly \(*\)-reversible.

**Corollary 12.** If \( T(R, R) \) is weakly reversible then \( R \) is weakly \(*\)-reversible.

**Corollary 13.** A commutative \(*\)-ring \( R \) is weakly \(*\)-reversible if and only if \( T(R, R) \), with adjoint involution, is weakly \(*\)-reversible.

From [13, Corollary 2.1], \( R \) is weakly IFP if and only if \( T(R, R) \) is weakly IFP and by Proposition 17, we have the following corollaries.

**Corollary 14.** If \( R \) is weakly IFP then \( T(R, R) \) is weakly quasi \(*\)-IFP.

**Corollary 15.** If \( T(R, R) \) is weakly IFP then \( R \) is weakly quasi \(*\)-IFP.

**Corollary 16.** A commutative \(*\)-ring \( R \) is weakly quasi \(*\)-IFP if and only if \( T(R, R) \), with adjoint involution, is weakly quasi \(*\)-IFP.

5 Extensions of \(*\)-Reversible and Weakly quasi-*\*-IFP \(*\)-Rings

In this section, the properties of \(*\)-reversible, central \(*\)-reversible and weakly quasi-*\*-IFP are shown to be extended from \(*\)-ring to its localization, polynomial, Laurent polynomial, Dorroh extension and from Ore \(*\)-ring to its classical
On Reversibility of Rings with involution

Let $R$ be a *-ring and $S$ be a multiplicatively closed subset of $R$ consisting of nonzero central regular elements, then the localization of $R$ to $S$ is $S^{-1}R = \{u^{-1}a|u \in S, a \in R\}$ is a *-ring with involution $\phi$ defined as:

$$(u^{-1}a)^{\phi} = u^{-1}a^{\ast} = u^{\ast-1}a^{\ast}.$$  

**Proposition 18.** A *-ring $R$ is *-reversible if and only if $S^{-1}R$ is *-reversible.

**Proof.** Let $R$ be a *-reversible *-ring and $\alpha \beta = 0 = \alpha \beta^{\ast}$ with $\alpha = u^{-1}a$, $\beta = v^{-1}b$ where $a, b \in R$ and $u, v \in S$. Hence $\alpha \beta = u^{-1}av^{-1}b = u^{-1}v^{-1}ab = (vu)^{-1}ab = 0$ and $\alpha \beta^{\ast} = u^{-1}a(v^{\ast})^{-1}b^{\ast} = u^{-1}(v^{\ast})^{-1}ab^{\ast} = (v^{\ast}u)^{-1}ab^{\ast} = 0$, since $S$ is contained in the center of $R$, so $ab = 0 = ab^{\ast}$. By hypothesis $ba = 0$ which implies $\beta \alpha = v^{-1}bu^{-1}a = v^{-1}u^{-1}ba = (uv)^{-1}ba = 0$ and $S^{-1}R$ is *-reversible. The converse is clear. \(\square\)

By a similar proof, we get analogous results for central *-reversible and weakly quasi-*-IFP *-rings.

**Proposition 19.** A *-ring $R$ is central *-reversible if and only if $S^{-1}R$ is central *-reversible.

**Proposition 20.** A *-ring $R$ is weakly quasi-*-IFP, if and only if $S^{-1}R$ is weakly quasi-*-IFP.

From Propositions 18, 19 and 20 we get the following corollaries.

**Corollary 17.** If $R$ is a reversible *-ring, then $S^{-1}R$ is *-reversible.

**Corollary 18.** If $S^{-1}R$ is a reversible *-ring, then $R$ is *-reversible.

**Corollary 19.** If $R$ is a central reversible *-ring, then $S^{-1}R$ is central *-reversible.

**Corollary 20.** If $S^{-1}R$ is a central reversible *-ring, then $R$ is central *-reversible.

**Corollary 21.** If $R$ has quasi-*-IFP, then $S^{-1}R$ is weakly quasi-*-IFP.

**Corollary 22.** If $S^{-1}R$ has quasi-*-IFP, then $R$ is weakly quasi-*-IFP.

The *-ring of Laurent polynomials in $x$, with coefficients in a *-ring $R$, consists of all formal sum $f(x) = \sum_{n=k}^{\infty} a_{i}x^{i}$ with obvious addition and multiplication, where $a_{i} \in R$ and $k, n$ are (possibly negative) integers and with involution $*$ defined as $f^{\ast}(x) = \sum_{i=k}^{\infty} a_{i}^{\ast}x^{i}$. We denote this ring as usual by $R[x; x^{-1}]$. 
Corollary 23. Let $R$ be a *-ring. Then $R[x]$ is *-reversible if and only if $R[x;x^{-1}]$ is *-reversible.

Proof. By [3, Proposition 3.15], it suffices to establish necessity. Clearly $S = \{1, x, x^2, \ldots \}$ is a multiplicatively closed subset of $R[x]$. Since $R[x;x^{-1}] = S^{-1}R[x]$, it follows that $R[x;x^{-1}]$ is *-reversible, by Proposition 18. □

Corollary 24. Let $R$ be a *-ring. Then $R[x]$ is central *-reversible if and only if $R[x;x^{-1}]$ is central *-reversible.

Proof. By Proposition 5, it suffices to prove necessity which can be done as the proof of Corollary 23 using Proposition 19. □

Corollary 25. For a *-ring, $R[x]$ is weakly quasi-*-IFP if and only if $R[x;x^{-1}]$ is weakly quasi-*-IFP.

Proof. By Proposition 16, it suffices to establish necessity which can be done as the proof of Corollary 23 using Proposition 20. □

From Corollary 25 we have the following results.

Corollary 26. If $R[x]$ has quasi-*-IFP, then $R[x;x^{-1}]$ is weakly quasi-*-IFP.

Corollary 27. If $R[x;x^{-1}]$ has quasi-*-IFP, then $R[x]$ is weakly quasi-*-IFP.

Corollary 28. If $R[x]$ has IFP, then $R[x;x^{-1}]$ is weakly quasi-*-IFP.

Corollary 29. If $R[x;x^{-1}]$ has IFP, then $R[x]$ is weakly quasi-*-IFP.

Corollary 30. If $R[x]$ has *-IFP, then $R[x;x^{-1}]$ is weakly quasi-*-IFP.

Corollary 31. If $R[x;x^{-1}]$ has *-IFP, then $R[x]$ is weakly quasi-*-IFP.

A *-ring $R$ is called a *-Armendariz *-ring if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = f(x)g^*(x) = 0$, then $a_i b_j = 0$ for all $i, j$. Consequently $a_i b_j^* = 0$.

Theorem 3. Let $R$ be a *-Armendariz *-ring. Then the following statements are equivalent.

1. $R$ is *-reversible (central *-reversible).
2. $R[x]$ is *-reversible (central *-reversible).
3. $R[x;x^{-1}]$ is *-reversible (central *-reversible).
Proof.

(1) $\implies$ (2): Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ with $f(x)g(x) = 0 = f(x)g^*(x)$. Since $R$ is *-Armendariz, $a_i b_j = 0 = a_i b_j^*$ for each $i$ and $j$. But $R$ is *-reversible (central *-reversible), hence $b_j a_i = 0$ ($b_j a_i$ is central) for each $i$ and $j$. It follows that $g(x)f(x) = 0$ ($g(x)f(x)$ is central) and $R[x]$ is *-reversible (central *-reversible).

(2) $\implies$ (1): Clear from [3, Proposition 3.15] (Proposition 5).

(2) $\iff$ (3): Follows from Corollary 23 (Corollary 24).

The following corollary is an immediate from Theorem 3.

**Corollary 32.** Let $R$ be an Armendariz *-ring. Then the following statements are equivalent.

1. $R$ is *-reversible (central *-reversible).
2. $R[x]$ is *-reversible (central *-reversible).
3. $R[x; x^{-1}]$ is *-reversible (central *-reversible).

The Dorroh extension $D(R, \mathbb{Z}) = \{(r, n) : r \in R, n \in \mathbb{Z}\}$ of a *-ring $R$ is a ring with componentwise addition and multiplication $(r_1, n_1)(r_2, n_2) = (r_1 r_2 + n_1 r_2 + n_2 r_1, n_1 n_2)$. The involution of $R$ can be extended naturally to $D(R, \mathbb{Z})$ as $(r, n)^* = (r^*, n)$ (see [2]). We have the following:

**Proposition 21.** A *-ring $R$ is *-reversible if and only if its Dorroh extension $D(R, \mathbb{Z})$ of $R$ is *-reversible.

**Proof.** The sufficiency is clear. For necessity, let $(r_1, n_1), (r_2, n_2) \in D(R, \mathbb{Z})$ with $(r_1, n_1)(r_2, n_2) = 0 = (r_1, n_1)(r_2, n_2)$, then from $0 = (r_1, n_1)(r_2, n_2) = (r_1 r_2 + n_1 r_2 + n_2 r_1, n_1 n_2)$ and $0 = (r_1, n_1)(r_2, n_2) = (r_1 r_2^* + n_1 r_2^* + n_2 r_1, n_1 n_2)$, we have $r_1 r_2 + n_1 r_2 + n_2 r_1 = 0$, $r_1 r_2^* + n_1 r_2^* + n_2 r_1 = 0$ and $n_1 n_2 = 0$. Since $\mathbb{Z}$ is *-domain, $n_1 = 0$ or $n_2 = 0$. If $n_1 = 0$, we get $0 = r_1 r_2 + n_2 r_1 = r_1(r_2 + n_2)$ and $0 = r_1 r_2^* + n_2 r_1 = r_1(r_2^* + n_2)$. From the *-reversibility of $R$ it follows that $0 = (r_2 + n_2)r_1 = r_2 r_1 + n_2 r_1 = (r_2, n_2)(r, 0)$ and so $D(R, \mathbb{Z})$ is *-reversible. □

By a similar proof to the previous proposition, we get the following.

**Proposition 22.** A *-ring $R$ is central *-reversible if and only if its Dorroh extension $D(R, \mathbb{Z})$ of $R$ is central *-reversible.

Recall that a ring $R$ is called right Ore if given $a, b \in R$ with $b$ regular there exist $a_1, b_1 \in R$ with $b_1$ regular such that $ab_1 = ba_1$. Left Ore is defined
similarly and $R$ is Ore ring if it is both right and left Ore. For *-rings, right Ore implies left Ore and vice versa. It is a known fact that $R$ is Ore if and only if its classical quotient ring $Q$ of $R$ exists and for *-rings, * can be extended to $Q$ by $(a^{-1}b)^* = b^*(a^*)^{-1}$ (see [12, Lamme 4]).

**Theorem 4.** Let $R$ be an Ore *-ring and $Q$ be its classical quotient *-ring, then $R$ is *-reversible if and only if $Q$ is *-reversible.

**Proof.** The sufficiency is clear by [3, Proposition 3.15]. The proof of necessity is similar to that of [10, Theorem 2.6]. □

From [10, Theorem 2.6] and **Theorem 4**, we have the following corollaries.

**Corollary 33.** If $R$ is a reversible *-ring, then $Q$ is *-reversible.

**Corollary 34.** If $Q$ is a reversible *-ring, then $R$ is *-reversible.

**Corollary 35.** If $R$ is a *-reversible *-ring, then $Q$ is central *-reversible (weakly *-reversible).

**Corollary 36.** If $Q$ is a *-reversible *-ring, then $R$ is central *-reversible (weakly *-reversible).

**Conclusion**

Finally, we can state following implications in the class of rings with involution.

\[
\begin{array}{cccc}
\text{weakly IFP} & \uparrow & \text{central reversible} & \uparrow & \text{weakly reversible} \\
\text{reversible} & \downarrow & \text{central reversible} & \downarrow & \text{weakly reversible} \\
\text{* - reversible} & \downarrow & \text{central * - reversible} & \downarrow & \text{weakly * - reversible} \\
\text{quasi * - IFP} & \downarrow & \text{* - Abelian} & \downarrow & \text{weakly quasi * - IFP} \\
\end{array}
\]

**References**


On Reversibility of Rings with involution


