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ON REVERSIBILITY OF RINGS WITH INVOLUTION

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Abstract

Let R be a ring with involution *. We give the notion of central *-reversible *-rings which generalizes weakly *-reversible *-rings. Moreover, we introduce the class of weakly *-rings which is a generalization of central *-reversible *-rings and investigate their properties. Further, a generalization of the class of quasi-*-IFP *-rings is given; namely weakly quasi-*-IFP *-rings. Since every *-reversible *-ring is central *-reversible, we give sufficient conditions for central *-reversible, weakly *-reversible and weakly quasi-*-IFP *-rings to be *-reversible and some examples are given to illustrate these situations. Finally, we show that the properties of *-reversible, central *-reversible, weakly *-reversible and weakly quasi-*-IFP can be transfer to some extensions of the *-ring.

1 Introduction

Throughout this paper, a ring will always mean an associative ring with unity unless otherwise stated. A ring R is said to be *-ring if on R there is defined an involution *; that is an anti-isomorphism of order two. The right annihilator of the nonempty set A of R is denoted by $r_R(A)$ and the right *-annihilator of A is denoted by $r_{*R}(A) = \{x \in R \mid Ax = Ax^* = 0\}$. If there is no ambiguity, we omit the subsuffix R. A *-*ideal* (self-adjoint) I of R is an ideal closed under involution. A self adjoint idempotent; $e^2 = e = e^*$, is

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called projection. A nonzero element a of a *-ring R is called **-zero divisor* if $ab = 0 = a^*b$, for some nonzero element $b \in R$ and R is **-domain* if it has no nonzero *-zero divisors, from [6]. A *-ring R is said to be *Abelian* (**-Abelian)* if every idempotent (projection) of R is center. A *-ring R is reduced if it has no nonzero nilpotent elements. A ring R is called *semicommutative* or has *(IFP)* if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$ (equivalently $r(a)$ is an ideal of R for all $a \in R$) (see [10]). A *-ring R is said to have **-IFP* if for all $a, b \in R$, $ab = 0$ implies $aRb^* = 0$ (equivalently $r(a)$ is a *-ideal of R for all $a \in R$) (see [4]). From [13], recall a ring R is *weakly semicommutative* if for all $a, b \in R$, $ab = 0$ implies arb is a nilpotent element for each $r \in R$. By [7], a ring R is called *reversible* if for all $a, b \in R$, $ab = 0$ implies $ba = 0$. According to [3], a *-ring R is called **-reversible* if for all $a, b \in R$, $ab = 0 = ab^*$ implies $ba = 0$, and R has *quasi-*-IFP* if for all $a, b \in R$, $ab = ab^* = 0$ implies $aRb = 0$. From [5], an element a of a *-ring R is called **-nilpotent* if $a^m = (aa^*)^n = 0$, for some positive integers m and n. R is **-reduced* if it has no nonzero *-nilpotent elements. Following [9], a *-ring R is called *Baer* *-ring if the right annihilator of every nonempty subset of R is generated, as a right ideal, by a projection. By [5], a *-ring R is called **-Baer* *-ring if the *-right annihilator of every nonempty subset of R is generated, as a biideal, by a projection. From [8] a ring R is *central reversible* rings if for all $a, b \in R$, $ab = 0$ implies ba belongs to the center of R and a ring R is called *weakly reversible* if $ab = 0$ implies $Rbra$ is nil left ideal of R, for all $a, b, r \in R$, from [11]. The natural numbers and the integers will be denoted by $\mathbb{N}$ and $\mathbb{Z}$, respectively. $\mathbb{M}_n(R)$ will denote the full matrix ring of all $n \times n$ matrices over the ring R, while $T_n(R)$ ($T_{nE}(R)$) will denote the $n \times n$ upper triangular matrix ring (with equal diagonal elements) over R.

In this paper, we introduce central and weakly *-reversible *-rings, both are proper generalizations of *-reversible *-rings. Moreover, the class of weakly *-reversible *-rings contains strictly central *-reversible *-rings. We also prove that central *-reversible *-rings are *-Abelian and there exists a *-Abelian *-ring which is not central *-reversible. Clearly *-reversible *-rings are quasi-*-IFP and example is given to show that the converse is not true and another example shows that commutative weakly *-reversible *-rings do not necessarily have quasi-*-IFP. It is also shown that if R is a commutative *-ring, then $T_{nE}(R)$ is weakly *-reversible (weakly quasi-*-IFP) *-ring. Moreover, weakly quasi-*-IFP condition is given for *-rings which generalizes quasi-*-IFP. We show also that commutative weakly quasi-*-IFP *-rings may not be quasi-*-IFP. Moreover, for a *-Armendariz *-ring R, we prove that R is *-reversible (central *-reversible) if and only if the polynomial *-rings $R[x]$ is *-reversible (central *-reversible) if and only if the Laurent polynomial *-ring $R[x; x^{-1}]$ is *-reversible (central *-reversible). Furthermore, it is proved that R is *-reversible (central *-reversible) if and only if the Dorroh extension $D(R, \mathbb{Z})$ of

R is *-reversible (central *-reversible). Finally, the Ore *-ring R is shown to be *-reversible if and only if its classical quotient Q is *-reversible.

2 Central *-Reversible *-Rings

In this section, we introduce and study the class of central *-reversible *-rings, which is a generalization of *-reversible *-rings. We start by giving the main definition.

Definition. A *-ring R is called *central *-reversible* if for all $a, b \in R$, $ab = 0 = ab^*$ implies ba is central in R. Consequently, b^*a is central in R.

Clearly, a central reversible *-ring is central *-reversible and a *-reversible *-ring is central *-reversible. However, the next result shows that $T_{3E}(R)$, in general, is central *-reversible but not *-reversible.

Proposition 1. *Let R be a commutative *-ring, then the ring*

$$T_{3E}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

with involution defined as $\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$ *is central *-reversible *-ring.*

Proof. Let $x = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}$ and $y = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in T_{3E}(R)$. If $xy = 0 = xy^*$, then we have the following equations :

$$a_1a_2 = 0 \tag{1}$$

$$a_1b_2 + b_1a_2 = 0, \quad a_1d_2 + b_1a_2 = 0 \tag{2}$$

$$a_1c_2 + b_1d_2 + c_1a_2 = 0, \quad a_1c_2 + b_1b_2 + c_1a_2 = 0 \tag{3}$$

$$a_1d_2 + d_1a_2 = 0, \quad a_1b_2 + d_1a_2 = 0. \tag{4}$$

Hence $yx = \begin{pmatrix} 0 & 0 & b_2d_1 - b_1d_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$, is central and consequently $T_{3E}(R)$ is central *-reversible. On the other hand, $T_{3E}(R)$ is not *-reversible, since $yx \neq 0$, while the converse is clear from [3, Example 3.8]. □

In general, **Proposition 1** is not true for $n \geq 4$ which is clear from the following example.

Example 1. Consider the *-ring $T_{4E}(\mathbb{Z})$ with the involution * defined as:
$$\begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & a_{34} & a_{24} & a_{14} \\ 0 & a & a_{23} & a_{13} \\ 0 & 0 & a & a_{12} \\ 0 & 0 & 0 & a \end{pmatrix}.$$
The matrices $A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ satisfies $AB = 0 = AB^*$, but $BA = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is not central and so $T_{4E}(\mathbb{Z})$ is not central*-reversible.

It is clear that each central reversible is central *-reversible. However, the converse is true when the ring has *-IFP as shown in the next result.

Proposition 2. *Let R be a *-ring. If R is central *-reversible and has *-IFP, then R is central reversible.*

Proof. Obvious, since $ab = 0$, implies $aRb^* = 0$, by *-IFP property, and R is central reversible. □

Recall that a *-ring R is *-semiprime if and only if it is semiprime (see ([1])). Next, we give some particular conditions for a central *-reversible *-ring to be *-reversible.

Proposition 3. *A semiprime central *-reversible *-ring is *-reversible.*

Proof. Assume that R is a semiprime central *-reversible *-ring. If $ab = ab^* = 0$, then ba is central and consequently $baRba = 0$. Form semiprimeness, we get $ba = 0$ and so R is *-reversible. □

Proposition 4. *If R is a *-Baer and central *-reversible *-ring, then R is *-reversible.*

Proof. Let R be a *-Baer *-ring and $ab = 0 = ab^*$, then there exists a projection $e \in R$ such that $r_*(a) = eRe$. We have $ae = 0$ and $b = ebe = eb$, since $b \in r_*(a) = eRe$. Hence $ba = eba = bae = 0$, since ba is central, and so R is *-reversible. □

Since each Bear *-ring is *-Bear, we have the following corollary.

Corollary 1. *If R is a Baer and central *-reversible *-ring, then R is *-reversible.*

Furthermore, the class of central *-reversible *-rings is clearly closed under direct sums (with changeless involution) and under taking *-subrings by [3], since every *-reversible *-ring is central *-reversible.

Proposition 5. *The class of central *-reversible *-ring is closed under direct sums and under taking *-subrings.*

Proposition 6. *Let R be a *-ring and e be a central projection of R. Then eR and $(1-e)R$ are *-reversible if and only if R is *-reversible.*

Proof. It suffices to show the necessity by [3, Proposition 3.15]. Let $ab = ab^* = 0$ with $a, b \,\epsilon\, R$, then $eab = eab^* = 0$ and $(1-e)ab = (1-e)ab^* = 0$. By assumption, we have $bea = 0$ and $b(1-e)a = 0$. Hence $ba = bea + [b(1-e)a] = 0$ and so R is *-reversible. □

By a similar proof as **Proposition 6**, and using **Proposition 5**, the following corollary is immediate.

Corollary 2. *Let R be a *-ring and e be a central projection of R. Then eR and $(1-e)R$ are central *-reversible if and only if R is central *-reversible.*

Recall that a *-ideal I of a *-ring R is **-nil* if each element of I is *-nilpotent.

Obviously, each *-nil ideal is nil. The following example shows that the converse is not always true.

Example 2. For the *-ring $R = \mathbb{M}_2(\mathbb{Z})$ of all 2×2 matrices over the integers $\mathbb{Z}$ with transpose of matrices as involution, the nonzero elements of the form $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ are all nilpotent but not *-nilpotent, since $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}^2 = 0$ but $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} = \begin{pmatrix} x^2 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$

We note that the homomorphic image of a cental *-reversible *-ring need not be central *-reversible as seen from the following example.

Example 3. Let D be a *-division ring, $R = D[x, y]$ and $I =< xy >$, where $xy \neq yx$. Since R is *-domain, R is central *-reversible. On the other hand, $(x+I)(y+I)$ and $(x+I)^*(y+I) = (x+I)(y+I)$ are both zero. But $(y+I)(x+I)$ is not central in R/I, hence R/I is not central *-reversible.

Moreover, the next example shows that if the homomorphic image of a *-ring R is central *-reversible,then R need not be central *-reversible.

Example 4. Let $R = \begin{pmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{pmatrix}$, where $\mathbb{F}$ is a field, with the adjoint involution * definition by $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$ for all $a, b, c \in \mathbb{F}$. Consider the *-ideal $I = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ of R. Then R/I is central *-reversible, because of the commutativity property of R/I. For $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$ where $B^* = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \in R$, we have $AB = 0 = AB^*$. Consider $C = \begin{pmatrix} c_1 & c_2 \\ 0 & c_3 \end{pmatrix} \in R$ with $c_1 \neq c_3$. It is clear that $CBA \neq BAC$ and therefore R is not central *-reversible.

Our next endeavour is to give a condition on the homomorphic image of a *-ring to be central *-reversible. Recall that a *-ring R is called *unit-central*, if all unit elements of R are central in R. Moreover, we show that every unit central *-ring is *-Abelian.

Proposition 7. *Let R be a unit-central*-ring. If I is a *-nil ideal of R, then R/I is central *-reversible.*

Proof. Let $a, b \in R$ with $(a+I)(b+I) = (a+I)(b+I)^* = I$. Then $ab \in I$, $ab^* \in I$ and so there exists a positive integers m, n, p and q such that $(ab)^m = 0$, $((ab)(ab)^*)^n = 0$, $(ab^*)^p = 0$ and $((ab^*)(ab^*)^*)^q = 0$. It follows that $(ba)^{m+1} = 0$, whence $1 - ba$ is unit and so central by hypothesis. Thus $rba = bar$ for any $r \in R$ and therefore $(b+I)(a+I)$ is central in R/I. □

Since each *-reversible *-ring is central *-reversible and each *-domain is *-reversible, by [3, Example 3.2], we have immediately the following corollary.

Corollary 3. *Every *-domain is a central *-reversible *-ring.*

The converse of **Corollary 3** is not true by **Example 4**. However, the converse is true for *-prime *-rings as follows.

Proposition 8. *Let R be a *-ring. Then R is *-prime and central *-reversible if and only if it is *-domain.*

Proof. Let R be *-prime and central *-reversible and $ab = ab^* = 0$ for some $a, b \in R$. We have $rab = rab^* = 0$ for every $r \in R$ and so bra and b^*ra are central. Since $bratb = 0$ and $bratb^* = 0$ for all $t \in R$, then $a = 0$ or $b = 0$ and R is a *-domain. The converse is obvious by **Corollary 3**. □

It is well known from [3, Corollary 3.7] that every *-reversible *-ring is *-Abelian. Similarly, we have the same result for central *-reversible case.

Proposition 9. *A central *-reversible *-ring R is *-Abelian.*

Proof. Let $e^2 = e = e^* \in R$. for any $r \in R$, $(re - ere)(1 - e) = (re - ere)(1 - e)^* = 0$ implies $(1 - e)(re - ere) = re - ere$ is central. Commuting $re-ere$ by e we get $re-ere = 0$. Similarly for any $r \in R$, $(r^*e-er^*e)(1-e) = (r^*e - er^*e)(1 - e)^* = 0$ implies $r^*e - er^*e = 0$. Therefore $re = ere = er$ and R is *-Abelian. □

The next example shows that the reverse implication of **Proposition 9** is not true in general; that is there exists a *-Abelian *-ring which is not central *-reversible, and hence is not *-reversible.

Example 5. The only projections of the *-ring
$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a \equiv c(mod\ 2), b \equiv 0(mod\ 2), a, b, c \in \mathbb{Z} \right\}$ under adjoint involution * are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and so R is *-Abelian. On the other hand, for
$x = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in R$ with $xy = xy^* = 0$, we have $yx = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$ is not central and so R is not central *-reversible.

3 Weakly *-Reversible *-Rings

In this section, we introduce another generalization for *-reversible; namely weakly *-reversible *-rings.

Definition. A *-ring R is called *weakly *-reversible* if for all $a, b, r \in R$, $ab = ab^* = 0$, implies $Rbra$ is a nil left (equivalently, $braR$ is a nil right) ideal of R. Consequently, Rb^*ra is a nil left (equivalently, b^*raR is a nil right) ideal of R.

Each commutative *-ring is weakly reversible. Clearly, each weakly reversible *-ring is weakly *-reversible. The converse is true when the ring has *-IFP as shown in the following.

Proposition 10. *Let R be a *-ring. If R is weakly *-reversible and has *-IFP, then R is weakly reversible.*

Proof. Obvious, since $ab = 0$, implies $aRb^* = 0$, by the *-IFP property, and R is weakly reversible. □

Moreover, we can easily prove the following result.

Proposition 11. *The class of weakly *-reversible *-ring is closed under direct sums (with changeless involution) and under taking *-subrings.*

Proposition 12. *For a commutative *-ring R, $T_{nE}(R)$ is a weakly *-reversible *-ring, with involution * defined by fixing the two diagonals considering the diagonal right / left lower as symmetric ones and interchange the symmetric elements about it; that is*

$$\begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2(n-1)} & a_{2n} \\ 0 & 0 & a & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_{(n-1)n} \\ 0 & 0 & 0 & 0 & \cdots & a \end{pmatrix}^* =$$

$$= \begin{pmatrix} a & a_{(n-1)n} & a_{(n-2)n} & \cdots & a_{2n} & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2(n-1)} & a_{1(n-1)} \\ 0 & 0 & a & \cdots & \cdots & a_{1(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_{12} \\ 0 & 0 & 0 & 0 & \cdots & a \end{pmatrix}$$

Proof. Let R be weakly *-reversible and $A = \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix}$, $B = \begin{pmatrix} b & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b & b_{23} & \cdots & b_{2n} \\ 0 & 0 & b & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix} \in T_{nE}(R)$ satisfy $AB = 0 = AB^*$. Hence $ab = 0 = ab^*$ and Since R is weakly *-reversible, then for $C = \begin{pmatrix} c & c_{12} & c_{13} & \cdots & c_{1n} \\ 0 & c & c_{23} & \cdots & c_{2n} \\ 0 & 0 & c & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}$ and $D = \begin{pmatrix} d & d_{12} & d_{13} & \cdots & d_{1n} \\ 0 & d & d_{23} & \cdots & d_{2n} \\ 0 & 0 & d & \cdots & d_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d \end{pmatrix} \in T_{nE}(R)$, there exists $k \in \mathbb{N}$, with $(cbda)^k = 0$. Thus

$(CBDA)^k = \begin{pmatrix} 0 & \bigstar & \bigstar & \cdots & \bigstar \\ 0 & 0 & \bigstar & \cdots & \bigstar \\ 0 & 0 & 0 & \cdots & \bigstar \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$ and $(CBDA)^{kn} = 0$ follows and $T_{nE}(R)$ is weakly *-reversible. [$\bigstar$ denotes an element of R] □

Next, the given example shows that there exists a weakly *-reversible and quasi *-IFP *-ring which is not *-reversible.

Example 6. Let R be a commutative *-ring. Then the *-ring

$$T_{3E}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\},$$

is weakly *-reversible by **Proposition 12**, for some $a \neq 0$. For $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have $AB = 0 = AB^*$ and $BA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so $T_{3E}(R)$ is not *-reversible, while it has quasi-*-IFP.

We note that if R is a commutative then the *-ring.

$$T_{nE}(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R, n \geq 3 \right\},$$

is not *-reversible by [3, Example 3.8] and is weakly *-reversible by **Proposition 12**. Moreover, it is clear that $T_{4E}(R)$ is not quasi-*-IFP and so $T_{nE}(R)$ is not quasi-*-IFP for $n \geq 4$.

The next example demonstrates that the condition $T_{nE}(R)$ in **Proposition 12**, cannot be weakened to the full matrix *-ring $\mathbb{M}_n(R)$, where n is any integer bigger than 1.

Example 7. Let R be a weakly *-reversible *-ring and n any integer bigger than 1, then $\mathbb{M}_2(R)$, with adjoint involution, is not weakly *-reversible. For $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $AB = 0 = AB^*$ and for $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2(R)$, we see that $RBCA = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$ is not nil.

The following result shows that the class of central *-reversible *-rings lies properly between the classes of *-reversible and weakly *-reversible *-rings.

Theorem 1. *Let R be a *-ring and consider the following conditions.*

1. *R is *-reversible.*
2. *R is central *-reversible.*

3. *R is weakly *-reversible.*
   Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

Proof.

(1) $\Longrightarrow$ (2)**:** Clearly.

(2) $\Longrightarrow$ (3)**:** Let $a, b \in R$ with $ab = ab^* = 0$. Then for all $s \in R$, $sab = sab^* = 0$ and bsa is central, since R is central *-reversible. Hence $(rbsa)^2 = (rbsa)(rbsa) = r(bsa)r(bsa) = rr(bs(ab)sa) = 0$, for all $r, s \in R$ and R is weakly *-reversible.

□

The converse of **Theorem 1** is not true by **Examples** 1 and 6. However, from **Corollary 3** and **Theorem 1** we get the following corollary.

Corollary 4. *Every *-domain is a weakly *-reversible *-ring.*

4 Weakly quasi-*-IFP

Here, weakly quasi-*-IFP *-rings are introduced as generalization for the class of quasi-*-IFP *-rings. First, we introduce weakly *-IFP *-rings.

Definition. A *-ring R is called *weakly *-IFP* if for all $a, b \in R$, $ab = 0$ implies $arb^* \in \text{nil}(R)$ for all $r \in R$.

Each commutative *-ring is weakly *-IFP. As before, one can easily prove the following result.

Proposition 13. *The class of weakly *-IFP *-ring is closed under direct sums (with changeless involution) and under taking *-subrings.*

Proposition 14. *For a commutative *-ring R, $T_{nE}(R)$ is weakly *-IFP, with involution * given in **Proposition 12**.*

Proof. Let $A = (a_{ij})$ and $B = (b_{ij}) \in T_{nE}(R)$ with $AB = 0$, where $1 \leq i \leq j \leq n$, then we have $ab = 0$, where a and b are the diagonal elements of A and B, respectively. Since R is weakly *-IFP, there exists $k \in \mathbb{N}$ such that $(acb)^k = 0$ for all $C = (c_{ij}) \in T_{nE}(R)$, where c is the diagonal element of C . Hence $((ACB^*)^k)^n = 0$ and $T_{nE}(R)$ is weakly *-IFP. □

It is clear that every *-ring having *-IFP is weakly *-IFP while the converse is not always true as shown by the following example.

Example 8. The *-ring $T_{3E}(\mathbb{Z})$ with the involution * given by: $\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$ is weakly *-IFP by **Proposition 14**. For $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have $AB = 0$ and $ARB^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$, so $T_{3E}(\mathbb{Z})$ has not *-IFP.

By the way, there exists a weakly IFP *-ring which is not weakly *-IFP as in the next example.

Example 9. Let $\mathbb{F}$ be a field and consider the *-ring $R = \mathbb{F} \bigoplus \mathbb{F}$, with the exchange involution $(a,b)^* = (b,a)$, for all $a, b \in \mathbb{F}$. R is clearly weakly IFP and is not weakly *-IFP.

Next, we define weakly quasi-*-IFP *-rings

Definition. A *-ring R is said to be *weakly quasi-*-IFP* if for all $a, b \in R$, $ab = 0 = ab^*$ implies arb is a nilpotent element for each $r \in R$. Consequently arb^* is also nilpotent.

Each commutative *-ring is weakly quasi *-IFP. Clearly, each weakly IFP *-ring is weakly quasi-*-IFP. The converse is true when the ring has *-IFP as shown in the following.

Proposition 15. *Let R be a *-ring. If R is weakly quasi-*-IFP and has *-IFP, then R is weakly IFP.*

Proof. Clearly, since $ab = 0$, implies $aRb^* = 0$, by the *-IFP property, and R is weakly quasi-*-IFP. □

Moreover, the class of weakly quasi-*-IFP *-ring is closed under direct sums (using changeless involution) and under taking *-subrings.

Proposition 16. *The class of weakly quasi-*-IFP *-ring is closed under direct sums and under taking *-subrings.*

By a proof similar to **Proposition 12**, we get the following.

Proposition 17. *If R is a commutative *-ring, then $T_{nE}(R)$ is weakly quasi-*-IFP, with involution * given in **Proposition 12**.*

Note that if R is a commutative *-ring then the *-ring.

$$T_{nE}(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R, n \geq 3 \right\},$$

is not *-reversible by [3, Example 3.8] and is weakly quasi-*-IFP by **Proposition 17**. However, It is clearly that $T_{4E}(R)$ is not quasi-*-IFP and so $T_{nE}(R)$ is not quasi-*-IFP for $n \geq 4$.

The next example demonstrates that the condition $T_{nE}(R)$ in **Proposition 17**, cannot be weakened to the full matrix *-ring $\mathbb{M}_n(R)$, where $n > 1$.

Example 10. $\mathbb{Z}$ is weakly quasi-*-IFP *-ring with identical involution, while the *-ring $\mathbb{M}_2(\mathbb{Z})$ with adjoint involution * is not weakly quasi-*-IFP. Indeed, the matrices $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ satisfy $AB = 0 = AB^*$ and for $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{M}_2(R)$, we have $ACB = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ is not nilpotent.

It is well known that every *-reversible *-ring has quasi-*-IFP by [3, Proposition 3.6]. Next, we prove that central *-reversible *-rings are weakly quasi-*-IFP.

Theorem 2. *Let R be a *-ring and consider the following conditions.*

1. *R is *-reversible.*
2. *R is central *-reversible.*
3. *R is weakly quasi-*-IFP.*
   Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

Proof.

(1) $\Longrightarrow$ (2). Is clear.

(2) $\Longrightarrow$ (3). If $a, b \in R$ satisfy $ab = ab^* = 0$, then ba is central and $(arb)^2 = 0$. Hence arb is nilpotent for all $r \in R$ and R is weakly quasi-*-IFP.

□

The converse of **Theorem 2** is not true by **Examples** 1 and 6. Moreover, from **Corollary 3** and **Theorem 2** we have the following result.

Corollary 5. *Every *-domain is a weakly quasi-*-IFP *-ring.*

From **Proposition 4** we have immediately the following corollary.

Corollary 6. *If R is a *-Baer and central *-reversible *-ring, then R has quasi *-IFP.*

From [8, Proposition 2.20], if R is central reduced (that is every nilpotent element is central), then $T(R, R)$ is central reversible and from [3, Proposition 3.14], if R is *-reduced and *-reversible, then $T(R, R)$, with componentwise involution, is *-reversible. Accordingly, we have the following corollaries.

Corollary 7. *If the *-ring R is central reduced *-ring then $T(R, R)$ is central *-reversible.*

Corollary 8. *If the *-ring R is reduced then $T(R, R)$ is central *-reversible.*

Corollary 9. *If the *-ring R is *-reduced and *-reversible then $T(R, R)$, with componentwise involution, is central *-reversible.*

Corollary 10. *If the *-ring R is reduced and *-reversible then $T(R, R)$, with componentwise involution, is central *-reversible.*

By [11, Corollary 2.4], R is weakly reversible if and only if its trivial extension $T(R, R)$ is weakly reversible and from **Proposition 12**, we have the following corollaries.

Corollary 11. *If R is weakly reversible then $T(R, R)$ is weakly *-reversible.*

Corollary 12. *If $T(R, R)$ is weakly reversible then R is weakly *-reversible.*

Corollary 13. *A commutative *-ring R is weakly *-reversible if and only if $T(R, R)$, with adjoint involution, is weakly *-reversible.*

From [13, Corollary 2.1], R is weakly IFP if and only if $T(R, R)$ is weakly IFP and by **Proposition 17**, we have the following corollaries.

Corollary 14. *If R is weakly IFP then $T(R, R)$ is weakly quasi *-IFP.*

Corollary 15. *If $T(R, R)$ is weakly IFP then R is weakly quasi *-IFP.*

Corollary 16. *A commutative *-ring R is weakly quasi *-IFP if and only if $T(R, R)$, with adjoint involution, is weakly quasi *-IFP.*

5 Extensions of *-Reversible and Weakly quasi-*-IFP *-Rings

In this section, the properties of *-reversible, central *-reversible and weakly quasi-*-IFP are shown to be extended from *-ring to its localization, polynomial, Laurent polynomial, Dorroh extension and from Ore *-ring to its classical

Quotient.

Let R be a *-ring and S be a multiplicatively closed subset of R consisting of nonzero central regular elements, then the localization of R to S is $S^{-1}R = \{u^{-1}a | u \in S, a \in R\}$ is a *-ring with involution $\diamond$ defined as:

$$(u^{-1}a)^{\diamond} = u^{-1^*}a^* = u^{*-1}a^*.$$

Proposition 18. *A *-ring R is *-reversible if and only if $S^{-1}R$ is *-reversible.*

Proof. Let R be a *-reversible *-ring and $\alpha\beta = 0 = \alpha\beta^{\diamond}$ with $\alpha = u^{-1}a$, $\beta = v^{-1}b$ where $a, b \in R$ and $u, v \in S$. Hence $\alpha\beta = u^{-1}av^{-1}b = u^{-1}v^{-1}ab = (vu)^{-1}ab = 0$ and $\alpha\beta^{\diamond} = u^{-1}a(v^*)^{-1}b^* = u^{-1}(v^*)^{-1}ab^* = (v^*u)^{-1}ab^* = 0$, since S is contained in the center of R, so $ab = 0 = ab^*$. By hypothesis $ba = 0$ which implies $\beta\alpha = v^{-1}bu^{-1}a = v^{-1}u^{-1}ba = (uv)^{-1}ba = 0$ and $S^{-1}R$ is *-reversible. The converse is clear. □

By a similar proof, we get analogous results for central *-reversible and weakly quasi-*-IFP *-rings.

Proposition 19. *A *-ring R is central *-reversible if and only if $S^{-1}R$ is central *-reversible.*

Proposition 20. *A *-ring R is weakly quasi-*-IFP, if and only if $S^{-1}R$ is weakly quasi-*-IFP.*

From **Propositions 18, 19 and 20** we get the following corollaries.

Corollary 17. *If R is a reversible *-ring, then $S^{-1}R$ is *-reversible.*

Corollary 18. *If $S^{-1}R$ is a reversible *-ring, then R is *-reversible.*

Corollary 19. *If R is a central reversible *-ring, then $S^{-1}R$ is central *-reversible.*

Corollary 20. *If $S^{-1}R$ is a central reversible *-ring, then R is central *-reversible.*

Corollary 21. *If R has quasi-*-IFP, then $S^{-1}R$ is weakly quasi-*-IFP.*

Corollary 22. *If $S^{-1}R$ has quasi-*-IFP, then R is weakly quasi-*-IFP.*

The *-ring of Laurent polynomials in x, with coefficients in a *-ring R, consists of all formal sum $f(x) = \sum_{i=k}^{n} a_i x^i$ with obvious addition and multiplication, where $a_i \in R$ and k, n are (possibly negative) integers and with involution * defined as $f^*(x) = \sum_{i=k}^{n} a_i^* x^i$. We denote this ring as usual by $R[x; x^{-1}]$.

Corollary 23. *Let R be a *-ring. Then $R[x]$ is *-reversible if and only if $R[x;x^{-1}]$ is *-reversible.*

Proof. By [3, Proposition 3.15], it suffices to establish necessity. Clearly $S = \{1, x, x^2, \cdots\}$ is a multiplicatively closed subset of $R[x]$. Since $R[x;x^{-1}] = S^{-1}R[x]$, it follows that $R[x;x^{-1}]$ is *-reversible, by **Proposition 18**. $\square$

Corollary 24. *Let R be a *-ring. Then $R[x]$ is central *-reversible if and only if $R[x;x^{-1}]$ is central *-reversible.*

Proof. By **Proposition 5**, it suffices to prove necessity which can be done as the proof of **Corollary 23** using **Proposition 19**. $\square$

Corollary 25. *For a *-ring, $R[x]$ is weakly quasi-*-IFP if and only if $R[x;x^{-1}]$ is weakly quasi-*-IFP.*

Proof. By **Proposition 16**, it suffices to establish necessity which can be done as the proof of **Corollary 23** using **Proposition 20**. $\square$

From **Corollary 25** we have the following results.

Corollary 26. *If $R[x]$ has quasi-*-IFP, then $R[x;x^{-1}]$ is weakly quasi-*-IFP.*

Corollary 27. *If $R[x;x^{-1}]$ has quasi-*-IFP, then $R[x]$ is weakly quasi-*-IFP.*

Corollary 28. *If $R[x]$ has IFP, then $R[x;x^{-1}]$ is weakly quasi-*-IFP.*

Corollary 29. *If $R[x;x^{-1}]$ has IFP, then $R[x]$ is weakly quasi-*-IFP.*

Corollary 30. *If $R[x]$ has *-IFP, then $R[x;x^{-1}]$ is weakly quasi-*-IFP.*

Corollary 31. *If $R[x;x^{-1}]$ has *-IFP, then $R[x]$ is weakly quasi-*-IFP.*

A *-ring R is called *a *-Armendariz *-ring* if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = f(x)g^*(x) = 0$, then $a_i b_j = 0$ for all i, j. Consequently $a_i b_j^* = 0$.

Theorem 3. *Let R be a *-Armendariz *-ring. Then the following statements are equivalent.*

1. *R is *-reversible (central *-reversible).*
2. *$R[x]$ is *-reversible (central *-reversible).*
3. *$R[x;x^{-1}]$ is *-reversible (central *-reversible).*

Proof.

(1) $\Longrightarrow$ (2)**:** Let $f(x)=\sum_{i=0}^{m} a_i x^i$ and $g(x)=\sum_{j=0}^{n} b_j x^j \in R[x]$ with $f(x)g(x)=0=f(x)g^*(x)$. Since R is *-Armendariz, $a_ib_j=0=a_ib_j^*$ for each i and j. But R is *-reversible (central *-reversible), hence $b_ja_i=0$ (b_ja_i is central) for each i and j. It follows that $g(x)f(x)=0$ ($g(x)f(x)$ is central) and $R[x]$ is *-reversible (central *-reversible).

(2) $\Longrightarrow$ (1)**:** Clear from [3, Proposition 3.15] (**Proposition 5**).

(2) $\Longleftrightarrow$ (3)**:** Follows from **Corollary 23** (**Corollary 24**).

□

The following corollary is an immediate from **Theorem 3**.

Corollary 32. *Let R be an Armendariz *-ring. Then the following statements are equivalent.*

1. *R is *-reversible (central *-reversible).*
2. *$R[x]$ is *-reversible (central *-reversible).*
3. *$R[x;x^{-1}]$ is *-reversible (central *-reversible).*

The Dorroh extension $D(R,\mathbb{Z})=\{(r,n): r\in R, n\in\mathbb{Z}\}$ of a *-ring R is a ring with componentwise addition and multiplication $(r_1,n_1)(r_2,n_2)=(r_1r_2+n_1r_2+n_2r_1,n_1n_2)$. The involution of R can be extended naturally to $D(R,\mathbb{Z})$ as $(r,n)^*=(r^*,n)$ (see [2]). We have the following:

Proposition 21. *A *-ring R is *-reversible if and only if its Dorroh extension $D(R,\mathbb{Z})$ of R is *-reversible.*

Proof. The sufficiency is clear. For necessity, let (r_1,n_1), $(r_2,n_2)\in D(R,\mathbb{Z})$ with $(r_1,n_1)(r_2,n_2)=0=(r_1,n_1)(r_2^*,n_2)$, then from $0=(r_1,n_1)(r_2,n_2)=(r_1r_2+n_1r_2+n_2r_1,n_1n_2)$ and $0=(r_1,n_1)(r_2^*,n_2)=(r_1r_2^*+n_1r_2^*+n_2r_1,n_1n_2)$, we have $r_1r_2+n_1r_2+n_2r_1=0$, $r_1r_2^*+n_1r_2^*+n_2r_1=0$ and $n_1n_2=0$. Since $\mathbb{Z}$ is *-domain, $n_1=0$ or $n_2=0$. If $n_1=0$, we get $0=r_1r_2+n_2r_1=r_1(r_2+n_2)$ and $0=r_1r_2^*+n_2r_1=r_1(r_2^*+n_2)$. From the *-reversibility of R it follows that $0=(r_2+n_2)r_1=r_2r_1+n_2r_1=(r_2,n_2)(r_1,0)$ and so $D(R,\mathbb{Z})$ is *-reversible. □

By a similar proof to the previous proposition, we get the following.

Proposition 22. *A *-ring R is central *-reversible if and only if its Dorroh extension $D(R,\mathbb{Z})$ of R is central *-reversible.*

Recall that a ring R is called right Ore if given $a,b\in R$ with b regular there exist $a_1,b_1\in R$ with b_1 regular such that $ab_1=ba_1$. Left Ore is defined similarly and R is *Ore ring* if it is both right and left Ore. For *-rings, right Ore implies left Ore and vice versa. It is a known fact that R is Ore if and only if its classical quotient ring Q of R exists and for *-rings, * can be extended to Q by $(a^{-1}b)^* = b^*(a^*)^{-1}$ (see[12, Lamme 4]).

Theorem 4. *Let R be an Ore *-ring and Q be its classical quotient *-ring, then R is *-reversible if and only if Q is *-reversible.*

Proof. The sufficiency is clear by [3, Proposition 3.15]. The proof of necessity is similar to that of [10, Theorem 2.6]. □

From [10, Theorem 2.6] and **Theorem 4**, we have the following corollaries.

Corollary 33. *If R is a reversible *-ring, then Q is *-reversible.*

Corollary 34. *If Q is a reversible *-ring, then R is *-reversible.*

Corollary 35. *If R is a *-reversible *-ring, then Q is central *-reversible (weakly *-reversible).*

Corollary 36. *If Q is a *-reversible *-ring, then R is central *-reversible (weakly *-reversible).*

Conclusion

Finally, we can sate following implications in the class of rings with involution.

$$\begin{array}{ccccc}
 & & weakly\ IFP & & \\
 & & \Uparrow & & \\
reversible & \Longrightarrow & central\ reversible & \Longrightarrow & weakly\ reversible \\
\Downarrow & & \Downarrow & & \Downarrow \\
^* - reversible & \Longrightarrow & central\ ^* - reversible & \Longrightarrow & weakly\ ^* - reversible \\
\Downarrow & & \Downarrow & & \\
quasi -^* - IFP & \Longrightarrow & ^* - Abelian & & \\
\Downarrow & & & & \\
weakly\ quasi -^* - IFP & & & &
\end{array}$$

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