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CLIQUE DECOMPOSITIONS OF THE DISTANCE MULTIGRAPH OF THE CARTESIAN PRODUCT OF GRAPHS

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Abstract

The distance multigraph of a graph G is the multigraph having the same vertex set as G with $d_G(u,v)$ edges connecting each pair of vertices u and v, where $d_G(u,v)$ is the distance between vertices u and v in G. In this paper, we introduce a technique to construct a clique decomposition of the distance multigraph of the Cartesian product of two arbitrary graphs. Such a construction is accomplished through using clique decompostions of the distance multigraphs of the component graphs and mutually orthogonal Latin squares.

1 Introduction

The *distance* $d_G(u,v)$ between vertices u and v in a graph G is the number of edges in a shortest path connecting u and v. The *distance multigraph* $D(G)$ of a graph G is the multigraph having the same vertex set as G with $d_G(u,v)$ edges connecting each pair of vertices u and v. A *clique* is a set of pairwise adjacent vertices. For convenience, sometimes we refer to a clique as a complete subgraph on its vertices. We write $K_n\{v_1,\dots,v_n\}$ for an n-vertex clique on the vertex set $\{v_1,\dots,v_n\}$. A *biclique* is a complete bipartite graph. A *decomposition* of a graph G is a collection of subgraphs of G such that each edge

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of G belongs to exactly one subgraph in the collection. If all subgraphs in the collection are cliques (or bicliques), then it is called a *clique* (or *biclique*, respectively) *decomposition*. If each subgraph in the collection is isomorphic to H for some subgraph H of G, then the decomposition is called an H*-decomposition*. For instance, Figure 1 illustrates the distance multigraph $D(C_5)$ and a K_3-decomposition of $D(C_5)$.

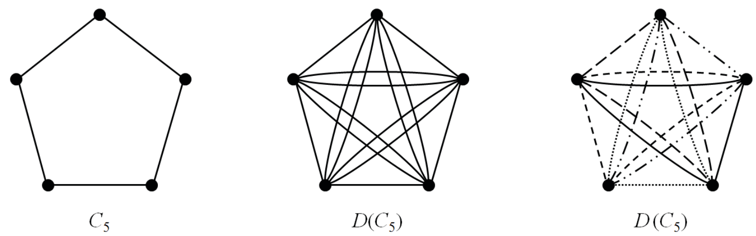

Figure 1: A K_3-decomposition of the distance multigraph $D(C_5)$

Unless we state multigraphs explicitly, our graphs are simple, finite, and connected. The *Cartesian product* of graphs G_1 and G_2, written $G_1 \square G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if (1) $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or (2) $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. Figure 2 illustrates examples of the Cartesian product of graphs.

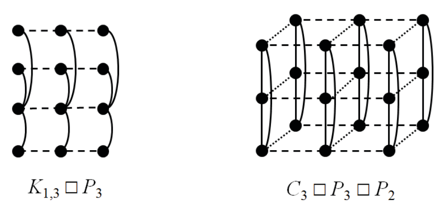

Figure 2: The graph $K_{1,3} \square P_3$ and the graph $C_3 \square P_3 \square P_2$

There are a wide range of applications of distance multigraphs in communication networks. For example, distance multigraphs can be used to solve the problem of finding a shortest route to transmit messages in a computer network. In this problem, a graph model can be produced by considering each computer as a vertex where any pair of vertices are adjacent if and only if messages can be transmitted between their corresponding computers. In 1971, Graham and Pollak [4] solved this kind of problem by devising an algorithm to label each vertex (formally says *address*) with a string of symbols from $\{0, 1, *\}$ of length

n. Furthermore, they require that the distance between each pair of vertices equals the number of positions in which one vertex has symbol 0 and the other one has symbol 1. Their goal is to minimize the length n of such addresses. Their approach is equivalent to finding a minimum biclique decomposition of the distance multigraph of the network graph; each position of an addressing represents one biclique in the decomposition in such a way that the vertices of symbols 0 and 1 form the two partite sets. In 1983, Winkler [8] proved that for a given graph G, there is always a biclique decomposition of $D(G)$ with less than $|V(G)|$ bicliques. In 2004, Elzinga et al. [3] showed that any biclique decompositions of the distance multigraph of the Petersen graph must have at least six bicliques. Analogously, the existence of a clique decomposition of the distance multigraph is equivalent to the existence of an addressing with a string of symbols from $\{0, 1\}$ to each vertex in such a way that the distance between each pair of vertices equals the number of positions in their respective addresses, which both equal 1. Each position of an addressing represents one clique in the decomposition formed simply by all vertices of symbol 1. For example, we can address the vertices $v_1, v_2, \ldots, v_5$ which are ordered around the cycle C_5 by $10101, 11010$, $01101, 10110$ and 01011. This yields a clique decomposition of $D(C_5)$, composing of five triangles $\{v_1, v_2, v_4\}$, $\{v_2, v_3, v_5\}$, $\{v_3, v_4, v_1\}$, $\{v_4, v_5, v_2\}$, $\{v_5, v_1, v_3\}$, respectively. Recently, in 2008, Cavers et al. [2] studied the problem of *clique partition number* (clique decomposition with minimum number of cliques) of distance multigraphs of a variety of graphs, namely, paths, cycles and complete multipartite graphs. In this paper, we introduce a technique to construct a clique decomposition of the distance multigraph of the Cartesian product of graphs in terms of clique decompositions of its original distance multigraphs. Interestingly, our results also yield an upper bound for the clique partition number of the distance multigraphs of the Cartesian product of some graphs such as odd cycles, the complete 3-partite graphs with partite sets of size 3^t, and Petersen graphs [Corollary 3.3].

A *Latin square* of order n is an $n \times n$ array of n symbols in which each symbol occurs exactly once in each row and in each column. Unless stated otherwise, the symbol set of a Latin square of order n is $\{1, 2, \ldots, n\}$. For Latin square L, we write L_{ij} for its entry in row i and column j. Two Latin squares L^1 and L^2 of the same order n are *orthogonal* if the n^2 ordered pairs resulting from superimposing the two Latin squares are distinct. A collection $\mathcal{L}$ of Latin squares is *mutually orthogonal* if all pairs of Latin squares in $\mathcal{L}$ are orthogonal. The size of a largest possible collection of mutually orthogonal Latin squares of order n is denoted $N(n)$. For brevity, we abbreviate mutually orthogonal Latin squares to MOLS. For any integer $n > 1$, we have $1 \leq N(n) \leq n-1$.(See more details in [7]). Built upon a series of research articles, it was finally proved in [1] that if $n \neq 2$ and $n \neq 6$, then $N(n) \geq 2$. The following three theorems regarding the existence of MOLS will be used to construct a clique decomposition of $D(G_1\square\cdots\square G_m)$ from clique decompositions of $D(G_1)$, ..., $D(G_m)$. We recall them here for future references.

Theorem 1.1. [1] *There exists a pair of orthogonal Latin squares of every order except* 2 *and* 6*.*

Theorem 1.2. [6] *If n is a prime power, then $N(n) = n - 1$.*

Theorem 1.3. [6] *If $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, where the p_i are distinct primes and each $a_i \geq 1$, then $\min\{N(p_i^{a_i}) : i = 1, 2, \dots, r\} \leq N(n) \leq n - 1$.*

In particular, $N(n_1 \times n_2) \geq \min\{N(n_1), N(n_2)\}$ for any positive integers n_1 and n_2.

2 A Clique Decomposition of $D(G_1\square\cdots\square G_m)$

The first lemma gives the number of edges in the distance multigraph of the Cartesian product of graphs. In order to prove our lemma, we use the following theorem which was proven in [5].

Theorem 2.1. [5] *Let G be the Cartesian product of graphs G_1, ..., G_m and let $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_m)$ be vertices in G. Then $d_G(u, v) = \sum_{i=1}^{m} d_{G_i}(u_i, v_i)$.*

Lemma 2.2. *Let G be the Cartesian product of graphs G_1, ..., G_m with $n_1, \dots, n_m$ vertices, respectively. Then $|E(D(G))| = \sum_{i=1}^{m} \alpha_i^2 |E(D(G_i))|$ where $\alpha_i = \prod_{k\in\{1,2,\dots,m\}\setminus\{i\}} n_k$.*

Proof. Theorem 2.1 yields the following result: for every vertex $u = (u_1, \dots, u_m) \in V(G)$, its degree in $D(G)$ can be computed by

$$\begin{aligned}
\deg_{D(G)}(u) &= \sum_{v\in V(G)} d_G(u, v) = \sum_{v=(v_1,\dots,v_m)\in V(G)} (\sum_{i=1}^{m} d_{G_i}(u_i, v_i)) \\
&= \sum_{i=1}^{m} \alpha_i \deg_{D(G_i)}(u_i).
\end{aligned}$$

Note that the last equality holds because G contains α_i copies of G_i, for $i = 1, 2, \dots, m$. Hence

$$\begin{aligned}
|E(D(G))| &= \frac{1}{2} \sum_{v\in V(G)} \deg_{D(G)}(v) = \frac{1}{2} \sum_{v\in V(G)} \sum_{i=1}^{m} \alpha_i \deg_{D(G_i)}(v_i) \\
&= \frac{1}{2} \sum_{i=1}^{m} (\alpha_i \sum_{v\in V(G)} \deg_{D(G_i)}(v_i)) \\
&= \sum_{i=1}^{m} (\alpha_i^2 \frac{1}{2} \sum_{v_i\in V(G_i)} \deg_{D(G_i)}(v_i)) = \sum_{i=1}^{m} \alpha_i^2 |E(D(G_i))|.
\end{aligned}$$

□

Our technique will use MOLS to generate cliques in a clique decomposition of $D(G_1\square\cdots\square G_m)$ from clique decompositions of $D(G_1)$, ..., $D(G_m)$. First, we will start with the case $m=2$. Note that throughout the paper any clique decomposition contains only cliques of minimum size two.

Theorem 2.3. *Given graphs G_1 and G_2 with n_1 and n_2 vertices, respectively. Let $\mathrm{P}(D(G_1))$ and $\mathrm{P}(D(G_2))$ be clique decompositions of $D(G_1)$ and $D(G_2)$, respectively. If the maximum size of cliques in $\mathrm{P}(D(G_1))$ and $\mathrm{P}(D(G_2))$ are at most $N(n_2)+2$ and $N(n_1)+2$, respectively, then there exists a clique decomposition of $D(G_1\square G_2))$ with $|\mathrm{P}(D(G_1))|n_2^2+|\mathrm{P}(D(G_2))|n_1^2$ cliques.*

Moreover, $D(G_1\square G_2)$ can be decomposed into $a_jn_2^2+b_jn_1^2$ copies of K_j for $j=2,3,\ldots,\max\{N(n_2)+2,N(n_1)+2\}$, where a_j and b_j are the number of copies of K_j in $\mathrm{P}(D(G_1))$ and $\mathrm{P}(D(G_2))$, respectively.

Proof. Let $V(G_1)=\{1,\ldots,n_1\}$ and $V(G_2)=\{1,\ldots,n_2\}$. Let $\{L^1,\ldots,L^{N(n_1)}\}$ be a set of $N(n_1)$ MOLS on the symbol set $\{1,2,\ldots,n_1\}$ and $\{S^1,\ldots,S^{N(n_2)}\}$ be a set of $N(n_2)$ MOLS on the symbol set $\{1,2,\ldots,n_2\}$. We create the clique decomposition of $G_1\square G_2$, denoted $\mathrm{P}(D(G_1\square G_2))$, by generating cliques from $\mathrm{P}(D(G_1))\cup\mathrm{P}(D(G_2))$ as follows.

Let $Q\in\mathrm{P}(D(G_1))\cup\mathrm{P}(D(G_2))$ and let $V(Q)=\{x_1,\ldots,x_s\}$. First, if $Q\in\mathrm{P}(D(G_1))$ then by the assumption s is at most $N(n_2)+2$. For all pairs $k,l\in\{1,2,\ldots,n_2\}$, include the clique $\{(x_1,k),(x_2,l),(x_3,S_{kl}^1),\ldots,(x_s,S_{kl}^{s-2})\}$ in $\mathrm{P}(D(G_1\square G_2))$. On the other hand, if $Q\in\mathrm{P}(D(G_2))$ then s is at most $N(n_1)+2$. Again, for all pairs $k,l\in\{1,2,\ldots,n_1\}$, include the clique $\{(k,x_1),(l,x_2),(L_{kl}^1,x_3),\ldots,(L_{kl}^{s-2},x_s)\}$ in $\mathrm{P}(D(G_1\square G_2))$.

By above, we get $\mathrm{P}(D(G_1\square G_2))$ which has $|\mathrm{P}(D(G_1))|n_2^2+|\mathrm{P}(D(G_2))|n_1^2$ cliques consisting of $a_jn_2^2+b_jn_1^2$ copies of K_j for $j=2,3,\ldots,\max\{N(n_2)+2,N(n_1)+2\}$. Therefore $\mathrm{P}(D(G_1\square G_2))$ can cover at most $|E(D(G_1))|n_2^2+|E(D(G_2))|n_1^2=|E(D(G_1\square G_2))|$ edges by Lemma 2.2.

Hence it suffices to show that each edge of $D(G_1\square G_2)$ is in a clique of $\mathrm{P}(D(G_1\square G_2))$. Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ be two distinct vertices in $D(G_1\square G_2)$. Note that there are $d_{G_1}(u_1,v_1)$ cliques in $\mathrm{P}(D(G_1))$ that contain both u_1 and v_1. Assume that Q is one such clique. Since $\{S^1,\ldots,S^{N(n_2)}\}$ is a set of MOLS, by our construction there always exists a clique in $\mathrm{P}(D(G_1\square G_2))$ generated from Q that contains both u and v. Therefore both u and v are together in at least $d_{G_1}(u_1,v_1)$ cliques in $\mathrm{P}(D(G_1\square G_2))$. Similarly, both u and v are together in at least $d_{G_2}(u_2,v_2)$ other cliques in $\mathrm{P}(D(G_1\square G_2))$. Hence, by Theorem 2.1, there are at least $d_{G_1\square G_2}(u,v)$ cliques in $\mathrm{P}(D(G_1\square G_2))$ that contain both u and v. □

To further illustrate the construction in Theorem 2.3, we give the following example.

Example 2.4. Let $\mathrm{P}(D(K_{1,3})) = \{K_4\{1,2,3,4\}, K_3\{2,3,4\}\}$ and $\mathrm{P}(D(P_3)) = \{K_3\{1,2,3\}, K_2\{1,3\}\}$ be clique decompositions of $D(K_{1,3})$ and $D(P_3)$ as shown in Figure 3. Here we write uv instead of a vertex (u,v) in $K_{1,3}\Box P_3$. We use a Latin square of order 4, namely, L^1 and two MOLS of order 3, namely, S^1 and S^2 as shown in Figure 4 to generate $\mathrm{P}(D(K_{1,3}\Box P_3))$, a clique decomposition of $D(K_{1,3}\Box P_3)$.

Therefore, we obtain $\mathrm{P}(D(K_{1,3}\Box P_3))$ as shown in Figure 5. The cliques in column A and B are generated by $K_4\{1,2,3,4\}$ of $\mathrm{P}(D(K_{1,3}))$ and $K_3\{2,3,4\}$ of $\mathrm{P}(D(K_{1,3}))$, respectively. Similarly, the cliques in column C and D are generated by $K_3\{1,2,3\}$ of $\mathrm{P}(D(P_3))$ and $K_2\{1,3\}$ of $\mathrm{P}(D(P_3))$, respectively.

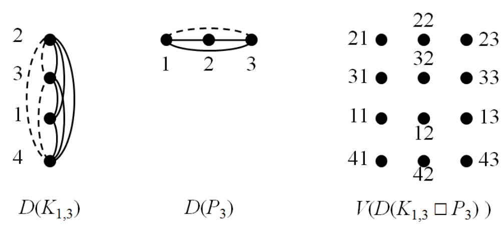

Figure 3: Clique decompositions of $D(K_{1,3})$ and $D(P_3)$

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

L^1

1	2	3
2	3	1
3	1	2

S^1

1	2	3
3	1	2
2	3	1

S^2

Figure 4: A Latin square of order 4 and two MOLS of order 3

The next theorem is a generalization of Theorem 2.3 to a distance multigraph of the Cartesian product of an arbitrary number of factors under certain assumptions.

Theorem 2.5. *Let G be the Cartesian product of graphs $G_1, \ldots, G_m$ with $n_1, \ldots, n_m$ vertices, respectively. For each $i = 1,2,\ldots,m$, let $\mathrm{P}(D(G_i))$ be a clique decomposition of $D(G_i)$. If the maximum size of cliques in $\mathrm{P}(D(G_i))$ is at most $\min_{k\in\{1,2,\ldots,m\}\setminus\{i\}}\{N(n_k)+2\}$, for all $i = 1,2,\ldots,m$, then there exists a clique decomposition of $D(G)$ with $\sum_{i=1}^{m}|\mathrm{P}(D(G_i))|\alpha_i^2$ cliques where $\alpha_i = \prod_{k\in\{1,2,\ldots,m\}\setminus\{i\}} n_k$.*

Proof. First note that the simple induction will not work since we lose control of the maximum size of cliques to go forward in the inductive step. Instead,

A	B	C	D
$K_4\{11,21,31,41\}$	$K_3\{21,31,41\}$	$K_3\{11,12,13\}$	$K_2\{11,13\}$
$K_4\{11,22,32,42\}$	$K_3\{21,32,42\}$	$K_3\{11,22,23\}$	$K_2\{11,23\}$
$K_4\{11,23,33,43\}$	$K_3\{21,33,43\}$	$K_3\{11,32,33\}$	$K_2\{11,33\}$
$K_4\{12,21,32,43\}$	$K_3\{22,31,42\}$	$K_3\{11,42,43\}$	$K_2\{11,43\}$
$K_4\{12,22,33,41\}$	$K_3\{22,32,43\}$	$K_3\{21,12,23\}$	$K_2\{21,13\}$
$K_4\{12,23,31,42\}$	$K_3\{22,33,41\}$	$K_3\{21,22,13\}$	$K_2\{21,23\}$
$K_4\{13,21,33,42\}$	$K_3\{23,31,43\}$	$K_3\{21,32,43\}$	$K_2\{21,33\}$
$K_4\{13,22,31,43\}$	$K_3\{23,32,41\}$	$K_3\{21,42,33\}$	$K_2\{21,43\}$
$K_4\{13,23,32,41\}$	$K_3\{23,33,42\}$	$K_3\{31,12,33\}$	$K_2\{31,13\}$
		$K_3\{31,22,43\}$	$K_2\{31,23\}$
		$K_3\{31,32,13\}$	$K_2\{31,33\}$
		$K_3\{31,42,23\}$	$K_2\{31,43\}$
		$K_3\{41,12,43\}$	$K_2\{41,13\}$
		$K_3\{41,22,33\}$	$K_2\{41,23\}$
		$K_3\{41,32,23\}$	$K_2\{41,33\}$
		$K_3\{41,42,13\}$	$K_2\{41,43\}$

Figure 5: All cliques in $\mathrm{P}(D(K_{1,3}\square P_3))$

we proceed by claiming that the clique decomposition of $D(G) = D((G_1\square G_2)\square G_3)\square\cdots\square G_m)$ can be constructed inductively by repeatedly using Theorem 2.3. Consider $D((G_1\square G_2)\square G_3)\square\cdots\square G_t)$ for $i = 2, 3, \ldots$. The basis step ($i = 2$) is just the result of Theorem 2.3. Assume the inductive step $i = m-1$. The key observation is that all the resulting cliques in the decomposition generated by the construction in Theorem 2.3 have size at most those of the original cliques. Since, by our assumption, for each $i = 1, 2, \ldots, m$, the maximum size of cliques in $\mathrm{P}(D(G_i))$ is at most $\min_{k\in\{1,2,\ldots,m\}\setminus\{i\}}\{N(n_k)+2\}$, we have that all cliques in the decomposition of $D(G_1\square G_2\square\cdots\square G_{m-1})$ have size no greater than $N(n_m)+2$. Moreover, the sizes of the cliques in $\mathrm{P}(D(G_m))$ are also at most $\min_{k\in\{1,2,\ldots,m-1\}}\{N(n_k)+2\} \le N(n_1n_2\ldots n_{m-1})+2$ by Theorem 1.3. We then can apply Theorem 2.3 again to have a clique decomposition $\mathrm{P}(D(G_1\square\cdots\square G_m))$ of size

$$|\mathrm{P}(D(G_1\square\cdots\square G_{m-1}))|n_m^2 + |\mathrm{P}(D(G_m))|(n_1n_2\cdots n_{m-1})^2.$$

By induction hypothesis, $|\mathrm{P}(D(G_1\square\cdots\square G_{m-1}))| = (\sum_{i=1}^{m-1}|\mathrm{P}(D(G_i))|\beta_i^2)n_m^2$ where $\beta_i = \prod_{k\in\{1,2,\ldots,m-1\}\setminus\{i\}} n_k$. Therefore

$$|\mathrm{P}(D(G_1\square\cdots\square G_m))| = \textstyle\sum_{i=1}^{m}|\mathrm{P}(D(G_i))|\alpha_i^2$$

where $\alpha_i = \prod_{k\in\{1,2,\ldots,m\}\setminus\{i\}} n_k$ as desired. $\square$

We can improve the result in Theorem 2.5 by raising the maximum limit of the clique sizes in the original clique decompositions. Theorem 1.3 guarantees that the maximum size of cliques in a clique decomposition of $D(G_i)$ in Theorem 2.6 is larger than those in Theorem 2.5.

Theorem 2.6. *Let G be the Cartesian product of graphs $G_1, \ldots, G_m$ with $n_1, \ldots, n_m$ vertices, respectively. For each $i = 1, 2, \ldots, m$, let $\mathrm{P}(D(G_i))$ be a clique decomposition of $D(G_i)$. If for each $i = 1, 2, \ldots, m$, the maximum size of cliques in $\mathrm{P}(D(G_i))$ is at most $N(\alpha_i)+2$, where $\alpha_i = \prod_{k\in\{1,2,\ldots,m\}\setminus\{i\}} n_k$, then there exists a clique decomposition of $D(G)$ with $\sum_{i=1}^{m} |\mathrm{P}(D(G_i))|\alpha_i^2$ cliques.*

Proof. We create the clique decomposition of $D(G)$, denoted $\mathrm{P}(D(G))$, by generating cliques from $\bigcup_{i=1}^{m} \mathrm{P}(D(G_i))$ as follows.

Let $Q \in \bigcup_{i=1}^{m} \mathrm{P}(D(G_i))$ and $V(Q) = \{x_1, \ldots, x_s\}$. Assume that $Q \in \mathrm{P}(D(G_t))$ for some $t \in \{1, 2, \ldots, m\}$. Then, $s \le N(\alpha_t) + 2$. Now let $\{L^1, \ldots, L^{N(\alpha_t)}\}$ be a set of $N(\alpha_t)$ MOLS on the symbol set $\{1, 2, \ldots, n_1\} \times \cdots \times \{1, 2, \ldots, n_{t-1}\} \times \{1, 2, \ldots, n_{t+1}\} \times \cdots \times \{1, 2, \ldots, n_m\}$.

Then the α_t^2 desired cliques generated from Q are given by

$$\begin{aligned}&\{(y_1^a, y_2^a, \ldots, y_{t-1}^a, x_a, y_{t+1}^a, \ldots, y_m^a) : a = 1, 2, \ldots, s-2\}\\ \cup&\{(k_1, k_2, \ldots, k_{t-1}, x_{s-1}, k_{t+1}, \ldots, k_m), (l_1, l_2, \ldots, l_{t-1}, x_s, l_{t+1}, \ldots, l_m)\},\end{aligned}$$

for $k = (k_1, k_2, \ldots, k_m) \in \{1, 2, \ldots, n_1\} \times \{1, 2, \ldots, n_2\} \times \cdots \times \{1, 2, \ldots, n_m\}$ and $l = (l_1, l_2, \ldots, l_m) \in \{1, 2, \ldots, n_1\} \times \{1, 2, \ldots, n_2\} \times \cdots \times \{1, 2, \ldots, n_m\}$, where $(y_1^a, y_2^a, \ldots, y_m^a) = L_{kl}^a$, for $1 \le a \le s-2$.

From the above construction, we obtain $\mathrm{P}(D(G))$ which has a total of $\sum_{i=1}^{m} \alpha_i^2\ |\mathrm{P}(D(G_i))|$ cliques consisting of $\sum_{i=1}^{m} a_j^{(i)}\alpha_i^2$ copies of K_j for $j = 2, 3, \ldots, \max_{i\in\{1,2,\ldots,m\}} \{N(\alpha_i) + 2\}$. It follows by Lemma 2.2 that $\mathrm{P}(D(G))$ can cover at most $\sum_{i=1}^{m} \alpha_i^2|E(D(G_i))| = |E(D(G))|$ edges.

Now it remains to show that each edge of $D(G)$ is in a clique of $\mathrm{P}(D(G))$. Let $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_m)$ be two distinct vertices in $D(G)$. By Theorem 2.1,there are $d_G(u, v) = \sum_{i=1}^{m} d_{G_i}(u_i, v_i)$ edges in $\mathrm{P}(D(G))$ connecting u and v. Let $i \in \{1, 2, \ldots, m\}$. By our construction, there are $d_{G_i}(u_i, v_i)$ cliques in $\mathrm{P}(D(G_i))$ that contain both u_i and v_i. Assume that Q is one such clique. Since $\{L^1, \ldots, L^{N(\alpha_i)}\}$ is a set of MOLS, there always exists a clique in $\mathrm{P}(D(G))$ generated from Q that contains both u and v. Therefore both u and v are together in $\sum_{i=1}^{m} d_{G_i}(u_i, v_i) = d_G(u, v)$ cliques in $\mathrm{P}(D(G))$ as desired. □

3 Additional Comments

The following corollary is the application of Theorem 2.6 together with Theorems 1.1 and 1.2 to some certain graph decompositions.

Corollary 3.1. *Let G be the Cartesian product of graphs $G_1, \ldots, G_m$. Then*

(i) *if $D(G_i)$ has a K_3-decomposition, for $i = 1, 2, \ldots, m$ then $D(G)$ also has a K_3-decomposition,*

(ii) *if $|V(G_i)| \neq 6$ and $D(G_i)$ has a K_4-decomposition, for $i = 1, 2, \ldots, m$ then $D(G)$ also has a K_4-decomposition,*

(iii) *if for all $i = 1, 2, \ldots, m$, $|V(G_i)|$ is a prime power and $D(G_i)$ has a K_s-decomposition where $s \leq \min_{j \in \{1,2,\ldots,m\}}\{|V(G_j)|\} + 1$ then $D(G)$ also has a K_s-decomposition.*

Now, we would like to point out the following interesting results, which essentially appear in [2], on the clique decompositions of some distance multigraphs.

Theorem 3.2. [2] (i) *The distance multigraph of an odd cycle has a K_3-decomposition.*

(ii) *For $t \in \mathbb{N}$, the distance multigraph of the complete multipartite graph $K_{3^t,3^t,3^t}$ has a $K_{2(3^t)}$-decomposition.*

(iii) *The distance multigraph of the Petersen graph has a K_6-decomposition.*

(iv) *The distance multigraph of the complete multipartite graph $K_{2,2,2}$ has a K_4-decomposition.*

(v) *Let G be the complete multipartite graph $K_{n_1,\ldots,n_k}$, where $n_1 + \cdots + n_k$ is a prime power and each partite set has size at least two. Then $D(G)$ can be decomposed into K_{n_1}, $\ldots$, K_{n_k} and $K_{n_1+\cdots+n_k}$.*

It should be emphasized that our construction technique gives a clique decomposition of the distance multigraph of the Cartesian product of graphs in terms of clique decompositions of its original distance multigraphs. Hence, by certain clique decompositions in Theorem 3.2 and our results, we obtain the following corollary.

Corollary 3.3. (i) *The distance multigraph of the Cartesian product of odd cycles has a K_3-decomposition.*

(ii) *For $t \in \mathbb{N}$, the distance multigraph of the Cartesian product of the complete multipartite graphs $K_{3^t,3^t,3^t}$ has a $K_{2(3^t)}$-decomposition.*

(iii) *The distance multigraph of the Cartesian product of at least three Petersen graphs has a K_6-decomposition.*

(iv) *The distance multigraph of the Cartesian product of at least three complete multipartite graphs $K_{2,2,2}$ has a K_4-decomposition.*

(v) *For $i = 1, 2, \ldots, m$, let G_i be a complete k-partite graph of p vertices where p is a prime power. Then $D(G_1 \Box \cdots \Box G_m)$ can be decomposed into copies of K_p together with copies of K_z for all sizes z of partite sets in each graph G_i for all i.*

Proof. (i) and (ii) follow directly from Theorem 3.2 and Corollary 3.1. (v) is straightforward by Theorem 2.5 and Theorem 1.2. To prove (iii) let G be the Cartesian product of m Petersen graphs where $m \geq 3$. So $|V(G)| = 10^m$. By Theorem 2.6, the maximum size of cliques in a clique decomposition of each distance multigraph of the Petersen graph that can be generated to a clique decomposition of $D(G)$ equals $N(100) + 2 = 8 + 2 = 10$ if $m = 3$, and equals $N(10^{m-1}) + 2 = N(2^{m-1}5^{m-1}) + 2 \geq \min\{2^{m-1} - 1, 5^{m-1} - 1\} + 2 \geq \min\{2^{4-1} - 1, 5^{4-1} - 1\} + 2 = 7 + 2 = 9$ if $m \geq 4$. However, this technique can not apply to the Cartesian product of two Petersen graphs because $N(10)+2 = 2 + 2 = 4$. (iv) follows similarly. $\square$

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