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NEW FIXED POINT THEOREMS IN METRIC TYPE SPACES

Doan Trong Hieu* and **Bui The Hung**†

* *Quang Ninh University of Industry, Viet Nam.*
e-mail: hieupci@gmail.com

† *Department of Mathematics,*
Thai Nguyen University of Education.
e-mail: hungbt.math@gmail.com

Abstract

In this paper, we prove new fixed point theorems for single valued and multivalued mappings in complete metric type spaces.

1 Introduction and Preliminaries

It is well known that the classical Banach fixed point principle plays an important role in applied mathematics. For instance, it is used to determine existence and uniqueness of solutions of differential and integral equations. In 1993 another axiom for semimetric spaces, which is weaker then the striangle inequality, was put forth by Czerwik [6] with a view of generalizing the Banach contraction mapping theorem.

Definition 1.1 ([6]). Let X be a nonempty set. A mapping $d : X \times X \to [0; +\infty)$ is called a *b-metric* on X if

(d1) $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leqslant 2[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then (X, d) is called a *b-metric space*.

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In 1998 Czerwik [7] generalized this notion where the constant 2 was replaced by a constant $s \geq 1$, also with the name b- metric. After that, in 2010 Khamsi and Hussain [8] reintroduced the notion of a b-metric under the name metric type.

Definition 1.2 ([8]). Let X be a nonempty set. A mapping $D : X \times X \to [0; +\infty)$ is called a *metric type* on X if

(D1) $D(x, y) = 0$ if and only if $x = y$;

(D2) $D(x, y) = D(y, x)$ for all $x, y \in X$;

(D3) $D(x, y) \leqslant K[D(x, z)+D(z, y)]$ for all $x, y, z \in X$ and for some constant $K > 0$.

Then (X, D, K) is called a *metric type space*.

Definition 1.3 ([8]). Let (X, D, K) be a metric type space and let $\{x_n\}$ be a sequence of points of X. We say that

(i) $\{x_n\}$ is convergent to $x \in X$ if $\lim\limits_{n\to\infty} D(x_n, x) = 0$.

(ii) $\{x_n\}$ is Cauchy sequence if $\lim\limits_{n,m\to\infty} D(x_n, x_m) = 0$.

(iii) (X, D, K) is complete if every Cauchy sequence of elements of X is convergent in X.

Subsequenly, many authors have studied fixed point theorems for metric type spaces, see for instance [1], [3], [4], [5], [6], [7], [8], [9]. In this paper we first prove some fixed point theorems for single valued and multivalued mappings in complete metric type spaces, and give several examples.

2 Fixed point theorem for a single valued mapping

In this section, we present two fixed point theorems for single valued mappings.

Theorem 2.1. *Let (X, D, K) is a complete metric type spaces and T be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$D(Tx, Ty) \leq \Big[\frac{D(x, Ty) + D(Tx, y) + D(x, y)}{(K + 1)D(x, Tx) + KD(y, Ty) + \alpha}\Big].D(x, y),$$

for all $x, y \in X$. Then

(1) *T has at least one fixed point $\bar{x} \in X$;*

(2) *for any $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point;*

(3) *if $\bar{x}, \bar{y} \in X$ are two distinct fixed points, then*

$$D(\bar{x}, \bar{y}) \geq \frac{\alpha}{3}.$$

Proof. Let $x_0 \in X$ be a fixed. Consider sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. Set $D_n = D(x_n, x_{n+1})$ for all $n \geq 0$. Then we deduce

$$\begin{aligned}
D_n &= D(Tx_{n-1}, Tx_n) \\
&\leq \Big[\frac{D(x_{n-1}, Tx_n) + D(Tx_{n-1}, x_n) + D(x_{n-1}, x_n)}{(K+1)D(x_{n-1}, Tx_{n-1}) + KD(x_n, Tx_n) + \alpha}\Big].D(x_{n-1}, x_n) \\
&= \Big[\frac{D(x_{n-1}, x_{n+1}) + D(x_{n-1}, x_n)}{(K+1)D(x_{n-1}, x_n) + KD(x_n, x_{n+1}) + \alpha}\Big].D(x_{n-1}, x_n) \\
&\leq \Big[\frac{(K+1)D(x_{n-1}, x_n) + KD(x_n, x_{n+1})}{(K+1)D(x_{n-1}, x_n) + KD(x_n, x_{n+1}) + \alpha}\Big].D(x_{n-1}, x_n) \\
&= \Big[\frac{(K+1)D_{n-1} + KD_n}{(K+1)D_{n-1} + KD_n + \alpha}\Big].D_{n-1} \text{ for all } n \geq 1.
\end{aligned}$$

Set

$$c_n = \frac{(K+1)D_{n-1} + KD_n}{(K+1)D_{n-1} + KD_n + \alpha} \text{ for all } n \geq 1.$$

Then $0 \leq c_n < 1$ and $D_n \leq c_n D_{n-1}$ for all $n \geq 1$. It follows that

$$D_n \leq D_{n-1} \text{ and } D_n \leq c_n c_{n-1} ... c_1 D_0 \text{ for all } n \geq 1.$$

By the function $f(t) = \frac{t}{t+\alpha}$ is increasing on $[0, +\infty)$, $c_n \leq c_{n-1}$ for all $n \geq 2$. Therefore

$$c_n c_{n-1} ... c_1 \leq c_1^n \to 0 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} c_n c_{n-1} ... c_1 = \lim_{n\to\infty} D_n = 0.$$

On the other hand, for all $n, p \geq 1$, we have

$$\begin{aligned}
D(x_n, x_{n+p}) &\leq KD(x_n, x_{n+1}) + K^2 D(x_{n+1}, x_{n+2}) + ... + K^p D(x_{n+p-1}, x_{n+p}) \\
&= KD_n + K^2 D_{n+1} + ... + K^p D_{n+p-1} \\
&\leq Kc_n c_{n-1} ... c_1 D_0 + K^2 c_{n+1} c_n ... c_1 D_0 + ... + K^p c_{n+p-1} c_{n+p-2} ... c_1 D_0 \\
&= (K + K^2 c_{n+1} + ... + K^p c_{n+p-1} c_{n+p-2} ... c_{n+1}) c_n c_{n-1} ... c_1 D_0 \\
&\leq (K + K^2 + ... + K^p) c_n c_{n-1} ... c_1 D_0 \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus
$$\lim_{n\to\infty} D(x_n, x_{n+p}) = 0 \text{ for all } p \geq 1.$$
This shows that $\{x_n\}$ be a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to some point $\bar{x} \in X$. We claim that $\bar{x}$ is a fixed point of T. Note that

$$\begin{aligned}
D(x_{n+1}, T\bar{x}) &= D(Tx_n, T\bar{x}) \\
&\leq [\frac{D(x_n, T\bar{x}) + D(Tx_n, \bar{x}) + D(x_n, \bar{x})}{(K+1)D(x_n, Tx_n) + KD(\bar{x}, T\bar{x}) + \alpha}].D(x_n, \bar{x}) \\
&= [\frac{D(x_n, T\bar{x}) + D(x_n, \bar{x}) + D(x_{n+1}, \bar{x})}{(K+1)D(x_n, x_{n+1}) + KD(\bar{x}, T\bar{x}) + \alpha}].D(x_n, \bar{x}) \\
&\leq [\frac{(K+1)D(x_n, \bar{x}) + KD(\bar{x}, T\bar{x}) + D(x_{n+1}, \bar{x})}{(K+1)D(x_n, x_{n+1}) + KD(\bar{x}, T\bar{x}) + \alpha}].D(x_n, \bar{x}) \to 0
\end{aligned}$$

as $n \to \infty$. We get
$$\lim_{n\to\infty} x_{n+1} = T\bar{x}.$$
Hence, $T\bar{x} = \bar{x}$ holds, thus, $\bar{x}$ is a fixed point of T. If $\bar{y}$ is a fixed point of T with $\bar{x} \neq \bar{y}$, then

$$\begin{aligned}
D(\bar{x}, \bar{y}) &= D(T\bar{x}, T\bar{y}) \\
&\leq [\frac{D(\bar{x}, T\bar{y}) + D(T\bar{x}, \bar{y}) + D(\bar{x}, \bar{y})}{(K+1)D(\bar{x}, T\bar{x}) + KD(\bar{y}, T\bar{y}) + \alpha}].D(\bar{x}, \bar{y}) \\
&= \frac{3D^2(\bar{x}, \bar{y})}{\alpha}.
\end{aligned}$$

This implies
$$D(\bar{x}, \bar{y}) \geq \frac{\alpha}{3}.$$

□

Example 2.2. Let $X = \{0, 1, 2\}$ and let $D : X \times X \to [0, +\infty)$ by

$$D(0,0) = D(1,1) = D(2,2) = 0,$$

$$D(0,1) = D(1,0) = 1,$$

$$D(0,2) = D(2,0) = 2,$$

$$D(1,2) = D(2,1) = 4.$$

Then $(X, D, K = 4)$ is a complete metric type space.

Let $T : X \to X$ by $T0 = 0, T1 = 1$ and $T2 = 0$. For $\alpha = 3$, we have

$$\begin{aligned} 1 = D(T0, T1) &\leq [\frac{D(0, T1) + D(T0, 1) + D(0, 1)}{5D(0, T0) + 4D(1, T1) + 3}].D(0, 1) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} 1 = D(T1, T2) &\leq [\frac{D(1, T2) + D(T1, 2) + D(1, 2)}{5D(1, T1) + 4D(2, T2) + 3}].D(1, 2) \\ &= \frac{36}{11}. \end{aligned}$$

Therefore T satisfies all the conditions of Theorem 2.1 for $\alpha = 3$. Also, T has two distinct fixed points $\{0, 1\}$ and

$$D(0, 1) = 1 \geq \frac{\alpha}{3} = 1.$$

Remark 2.1. Note that in Theorem 2.1 the ration

$$M(x, y) := \frac{D(x, Ty) + D(Tx, y) + D(x, y)}{(K + 1)D(x, Tx) + KD(y, Ty) + \alpha}$$

might be greater or less than 1. In Example 2.2, we have $M(x, y) \leq 1$ for all $x, y \in X$. The following example shows that $M(x, y) > 1$ for all $x \neq y \in X$.

Example 2.3. Let $X = \{0, 1, 2\}$ and let $D : X \times X \to [0, +\infty)$ by

$$D(0, 0) = D(1, 1) = D(2, 2) = 0,$$

$$D(0, 1) = D(1, 0) = \frac{1}{2},$$

$$D(0, 2) = D(2, 0) = 1,$$

$$D(1, 2) = D(2, 1) = 2.$$

Then $(X, D, K = \frac{4}{3})$ is a complete metric type space.

Let $T : X \to X$ by $T0 = 0, T1 = 1$ and $T2 = 2$. For $\alpha = 1$, we have

$$\frac{D(0, T1) + D(T0, 1) + D(0, 1)}{(\frac{4}{3} + 1)D(0, T0) + \frac{4}{3}D(1, T1) + 1} = \frac{3}{2},$$

$$\frac{D(1,T2)+D(T1,2)+D(1,2)}{(\frac{4}{3}+1)D(1,T1)+\frac{4}{3}D(2,T2)+1}=6,$$

$$\frac{D(0,T2)+D(T0,2)+D(0,2)}{(\frac{4}{3}+1)D(0,T0)+\frac{4}{3}D(2,T2)+1}=3.$$

Hence $M(x,y)>1$ for all $x \neq y \in X$. On the other hand, we have

$$\frac{1}{2}=D(T0,T1)\leq M(0,1).D(0,1)=\frac{3}{4},$$

$$2=D(T1,T2)\leq M(1,2).D(1,2)=12,$$

$$1=D(T0,T2)\leq M(0,2).D(0,2)=3.$$

Therefore T satisfies all the conditions of Theorem 2.1 for $\alpha = 1$. Also, T has three distinct fixed points $\{0,1,2\}$ and

$$D(\bar{x},\bar{y})\geq\frac{\alpha}{3}=\frac{1}{3}\text{ for all }\bar{x}\neq\bar{y}\in X.$$

Theorem 2.4. *Let (X,D,K) is a complete metric type space and T be a single valued mapping from X to itself. Suppose there exists $\alpha>0$ such that*

$$D(Tx,Ty)\leq\Big[\frac{D(x,Ty)+D(Tx,y)+D(x,Tx)+D(y,Ty)+D(x,y)}{(K+2)D(x,Tx)+(K+1)D(y,Ty)+\alpha}\Big].D(x,y),$$

for all $x,y\in X$. Then

(1) *T has at least one fixed point $\bar{x}\in X$;*

(2) *for any $x\in X$, the sequence $\{T^n x\}$ converges to a fixed point;*

(3) *if $\bar{x},\bar{y}\in X$ are two distinct fixed points, then*

$$D(\bar{x},\bar{y})\geq\frac{\alpha}{3}.$$

Proof. Let $x_0\in X$ be a fixed. Consider sequence $\{x_n\}$ by $x_{n+1}=Tx_n$ for all $n\geq 0$. Set $D_n=D(x_n,x_{n+1})$ for all $n\geq 0$. Then we have

$$\begin{aligned}
D_n &= D(Tx_{n-1},Tx_n)\\
&\leq \Big[\frac{(K+2)D(x_{n-1},x_n)+(K+1)D(x_n,x_{n+1})}{(K+2)D(x_{n-1},x_n)+(K+1)D(x_n,x_{n+1})+\alpha}\Big].D(x_{n-1},x_n)\\
&= \Big[\frac{(K+2)D_{n-1}+(K+1)D_n}{(K+2)D_{n-1}+(K+1)D_n+\alpha}\Big].D_{n-1}\text{ for all }n\geq 1.
\end{aligned}$$

We put
$$c_n = \frac{(K+2)D_{n-1} + (K+1)D_n}{(K+2)D_{n-1} + (K+1)D_n + \alpha} \text{ for all } n \geq 1.$$
Then $0 \leq c_n < 1$ and $D_n \leq c_n D_{n-1}$ for all $n \geq 1$. By using an argument similar to that of the proof of Theorem 2.1, we have completes the proof.

□

Remark 2.2. Since
$$M(x,y) \leq \frac{D(x,Ty) + D(Tx,y) + D(x,Tx) + D(y,Ty) + D(x,y)}{(K+2)D(x,Tx) + (K+1)D(y,Ty) + \alpha},$$
for all $x, y \in X$ and $\alpha > 0$, where $M(x,y) = \frac{D(x,Ty)+D(Tx,y)+D(x,y)}{(K+1)D(x,Tx)+KD(y,Ty)+\alpha}$, then Theorem 2.4 implies Theorem 2.1.

3 Fixed point theorem for a multivalued mapping

Let (X, D, K) be an metric type space. Then for each $x \in X$ and $r > 0$, the set
$$B_D(x,r) := \{y \in X : D(x,y) < r\}$$
denotes the open r- ball at x with respect to D. Note that a subset U is open in (X, D, K) if and only if for each $x \in U$, there exists $r_x > 0$ such that
$$B_D(x, r_x) \subset U.$$
The topology $\mathcal{T}_D$ on X is the family consisting of all open subsets in (X, D, K).

For a sequence $\{x_n\}$ in (X, D, K),
$$\lim_{n\to\infty} x_n = x \in X \text{ if and only if } \lim_{n\to\infty} D(x_n, x) = 0.$$
Then the convergence on (X, D, K) induces the sequential topology $\mathcal{T}$ on X in the sense of An, Tuyen and Dung [2]. It is well-known that the topology $\mathcal{T}_D$ and the sequential topology $\mathcal{T}$ on a metric type space (X, D, K) are coincident (see, Proposition 3.3 in [2]). Then $\lim\limits_{n\to\infty} x_n = x$ in (X, D, K) if and only if $\lim\limits_{n\to\infty} x_n = x$ in $(X, \mathcal{T})$.

In the next, every metric type space (X, D, K) is always understood to be a topological space with the sequential topology $\mathcal{T}$. A subset F is closed in (X, D, K) if $X \backslash F$ is open in (X, D, K). A subset A in metric type space (X, D, K) is said to be bounded if A is contained in some ball $B_D(x, r)$ of (X, D, K).

Now, let $CB(X)$ be the collection of all nonempty bounded closed subsets of X. Let $T : X \to CB(X)$ be a multivalued mapping on X. Let H be the Hausdorff metric on $CB(X)$ induced by D, that is,

$$H(A, B) := \max\{\sup_{x\in B} d(x, A); \sup_{x\in A} d(x, B)\},$$

where $A, B \in CB(X)$ and $d(x, A) := \inf_{y\in A} D(x, y)$. Donote

$$\delta(x, A) := \sup_{y\in A} D(x, y).$$

Theorem 3.1. *Let (X, D, K) is a complete metric type space and let $T : X \to CB(X)$ be an multivalued mapping. Suppose there exists $\alpha > 0$ such that*

$$H(Tx, Ty) \leq \Big[\frac{d(x, Ty) + d(Tx, y) + D(x, y)}{(K + 1)\delta(x, Tx) + K\delta(y, Ty) + \alpha}\Big].D(x, y),$$

for all $x, y \in X$. Then

(1) *T has at least one fixed point $\bar{x} \in X$;*

(2) *if $\bar{x}, \bar{y} \in X$ are two fixed points, then*

$$D(\bar{x}, \bar{y}) \geq \sqrt{\frac{\alpha}{3} H(T\bar{x}, T\bar{y})}.$$

Proof. Let $x_0 \in X$ and choose $x_1 \in Tx_0$.

Step 1. If $H(Tx_0, Tx_1) = 0$ then $Tx_0 = Tx_1$. Thus, x_1 is a fixed point of T. If $H(Tx_0, Tx_1) > 0$, then for each $h_1 > 1$, there exists $x_2 \in Tx_1$ such that

$$D(x_1, x_2) < h_1 H(Tx_0, Tx_1).$$

Step 2. Similarly, if $H(Tx_1, Tx_2) = 0$ then $Tx_1 = Tx_2$. Thus, x_2 is a fixed point of T. If $H(Tx_1, Tx_2) > 0$, then for each $h_2 > 1$, there exists $x_3 \in Tx_2$ such that

$$D(x_2, x_3) < h_2 H(Tx_1, Tx_2).$$

...

Step n. Continuing in this manner, if $H(Tx_{n-1}, Tx_n) = 0$ then $Tx_{n-1} = Tx_n$. Thus, x_n is a fixed point of T. If $H(Tx_{n-1}, Tx_n) > 0$ then for each $h_n > 1$, there exists $x_{n+1} \in Tx_n$ such that

$$D(x_n, x_{n+1}) < h_n H(Tx_{n-1}, Tx_n).$$

We have

$$\begin{aligned} H(Tx_{n-1}, Tx_n) &\leq \big[\frac{d(x_{n-1}, Tx_n) + d(Tx_{n-1}, x_n) + D(x_{n-1}, x_n)}{(K+1)\delta(x_{n-1}, Tx_{n-1}) + K\delta(x_n, Tx_n) + \alpha}\big].D(x_{n-1}, x_n) \\ &= \big[\frac{d(x_{n-1}, Tx_n) + D(x_{n-1}, x_n)}{(K+1)\delta(x_{n-1}, Tx_{n-1}) + K\delta(x_n, Tx_n) + \alpha}\big].D(x_{n-1}, x_n). \end{aligned}$$

On the other hand, for some $y_n \in T(x_n)$, we have

$$\begin{aligned} \frac{d(x_{n-1}, Tx_n) + D(x_{n-1}, x_n)}{(K+1)\delta(x_{n-1}, Tx_{n-1}) + K\delta(x_n, Tx_n) + \alpha} &\leq \frac{D(x_{n-1}, y_n) + D(x_{n-1}, x_n)}{(K+1)D(x_{n-1}, x_n) + KD(x_n, y_n) + \alpha} \\ &\leq \frac{(K+1)D(x_{n-1}, x_n) + KD(x_n, y_n)}{(K+1)D(x_{n-1}, x_n) + KD(x_n, y_n) + \alpha}. \end{aligned}$$

Hence

$$H(Tx_{n-1}, Tx_n) \leq \big[\frac{(K+1)D(x_{n-1}, x_n) + KD(x_n, y_n)}{(K+1)D(x_{n-1}, x_n) + KD(x_n, y_n) + \alpha}\big].D(x_{n-1}, x_n).$$

Set

$$c_n = \frac{(K+1)D(x_{n-1}, x_n) + KD(x_n, y_n)}{(K+1)D(x_{n-1}, x_n) + KD(x_n, y_n) + \alpha}.$$

Then $0 < c_n < 1$ and

$$D_n < h_n c_n D_{n-1}, \text{ where } D_n = D(x_n, x_{n+1}), D_{n-1} = D(x_{n-1}, x_n).$$

We choose $h_n = \frac{1}{\sqrt{c_n}}$. Then we have

$$D_n < \sqrt{c_n} D_{n-1}.$$

This implies

$$D_n < \sqrt{c_n c_{n-1} ... c_1} D_0.$$

By using an argument similar to that of the proof of Theorem 2.1, there exists $\bar{x} \in X$ such that $\lim_{n\to\infty} x_n = \bar{x}$. Now, we show that $\bar{x}$ is a fixed point of T. Indeed, we have

$$\begin{aligned} d(\bar{x}, T\bar{x}) &\leq KD(x_{n+1}, \bar{x}) + KH(Tx_n, T\bar{x}) \\ &\leq KD(x_{n+1}, \bar{x}) + K\big[\frac{d(x_n, T\bar{x}) + d(Tx_n, \bar{x}) + D(x_n, \bar{x})}{(K+1)\delta(x_n, Tx_n) + K\delta(\bar{x}, T\bar{x}) + \alpha}\big].D(x_n, \bar{x}) \\ &\leq KD(x_{n+1}, \bar{x}) + K\big[\frac{2D(x_n, \bar{x}) + d(\bar{x}, T\bar{x}) + D(x_{n+1}, \bar{x})}{(K+1)D(x_n, x_{n+1}) + K\delta(\bar{x}, T\bar{x}) + \alpha}\big].D(x_n, \bar{x}). \end{aligned}$$

On taking limit on both sides of above inequality, we have $d(\bar{x}, T\bar{x}) = 0$. It means that $\bar{x} \in T\bar{x}$. Now, if $\bar{y}$ is a fixed point of T, then we have

$$\begin{aligned} H(T\bar{x}, T\bar{y}) &\leq [\frac{d(\bar{x}, T\bar{y}) + d(T\bar{x}, \bar{y}) + D(\bar{x}, \bar{y})}{(K+1)\delta(\bar{x}, T\bar{x}) + K\delta(y, T\bar{y}) + \alpha}].D(\bar{x}, \bar{y}) \\ &\leq [\frac{D(\bar{x}, \bar{y}) + D(\bar{x}, \bar{y}) + D(\bar{x}, \bar{y})}{\alpha}].D(\bar{x}, \bar{y}) \end{aligned}$$

This implies

$$D(\bar{x}, \bar{y}) \geq \sqrt{\frac{\alpha}{3} H(T\bar{x}, T\bar{y})}.$$

□

Example 3.2. Let $X = \{0, 1, 2\}$ and let $D : X \times X \to [0, +\infty)$ by

$$D(x, y) = \begin{cases} 0, & \text{if } x = y \in X, \\ 2, & \text{if } x \neq y \in X. \end{cases}$$

Then $(X, D, K = 1)$ is a complete metric type space.

Let $T : X \to CB(X)$ by $T0 = \{0\}, T1 = \{1\}$ and $T2 = \{1, 2\}$. For $\alpha = 2$, we have

$$H(T0, T1) = H(T0, T2) = H(T1, T2) = 2,$$

and

$$[\frac{d(0, T1) + d(T0, 1) + D(0, 1)}{2\delta(0, T0) + \delta(1, T1) + 2}].D(0, 1) = [\frac{D(0, 1) + D(0, 1) + D(0, 1)}{2\delta(0, 0) + \delta(1, 1) + 2}].D(0, 1) = 6,$$

$$[\frac{d(0, T2) + d(T0, 2) + D(0, 2)}{2\delta(0, T0) + \delta(2, T2) + 2}].D(0, 2) = [\frac{D(0, 1) + D(0, 2) + D(0, 2)}{2\delta(0, 0) + \delta(2, \{1, 2\}) + 2}].D(0, 2) = 3,$$

$$[\frac{d(1, T2) + d(T1, 2) + D(1, 2)}{2\delta(1, T1) + \delta(2, T2) + 2}].D(1, 2) = [\frac{D(1, 1) + D(1, 2) + D(1, 2)}{2\delta(1, 1) + \delta(2, \{1\}) + 2}].D(1, 2) = 2.$$

Hence

$$H(T0, T1) \leq [\frac{d(0, T1) + d(T0, 1) + D(0, 1)}{2\delta(0, T0) + \delta(1, T1) + 2}].D(0, 1),$$

$$H(T0, T2) \leq [\frac{d(0, T2) + d(T0, 2) + D(0, 2)}{2\delta(0, T0) + \delta(2, T2) + 2}].D(0, 2),$$

$$H(T1, T2) \leq [\frac{d(1, T2) + d(2, T1) + D(1, 2)}{2\delta(1, T1) + \delta(2, T2) + 2}].D(1, 2).$$

Therefore T satisfies all the conditions of Theorem 3.1 for $\alpha = 2$. Also, T has three distinct fixed points $\{0, 1, 2\}$ and

$$D(\bar{x}, \bar{y}) \geq \sqrt{\frac{2}{3} H(T\bar{x}, T\bar{y})} \text{ for all } \bar{x}, \bar{y} \in \{0, 1, 2\}.$$

Theorem 3.3. *Let (X, D, K) is a complete metric type space and let $T : X \to CB(X)$ be an multivalued mapping. Suppose there exists $\alpha > 0$ such that*

$$H(Tx, Ty) \leq \Big[\frac{d(x, Ty) + d(Tx, y) + d(x, Tx) + d(y, Ty) + D(x, y)}{(K+2)\delta(x, Tx) + (K+1)\delta(y, Ty) + \alpha}\Big].D(x, y),$$

for all $x, y \in X$. Then

(1) *T has at least one fixed point $\bar{x} \in X$;*

(2) *if $\bar{x}, \bar{y} \in X$ are two fixed points, then*

$$D(\bar{x}, \bar{y}) \geq \sqrt{\frac{\alpha}{3} H(T\bar{x}, T\bar{y})}.$$

Proof. Let $x_0 \in X$. By using an argument similar to that of the proof of Theorem 3.1, for each $n \geq 1$, there exists $x_1, x_2, x_3, ..., x_n$ with $x_n \in Tx_{n-1}$ for all $n \geq 1$. If $H(Tx_{n-1}, Tx_n) = 0$ then $Tx_{n-1} = Tx_n$. Thus, x_n is a fixed point of T. If $H(Tx_{n-1}, Tx_n) > 0$ then for each $h_n > 1$, there exists $x_{n+1} \in Tx_n$ such that

$$D(x_n, x_{n+1}) < h_n H(Tx_{n-1}, Tx_n).$$

Then for some $y_n \in T(x_n)$, we have

$$H(Tx_{n-1}, Tx_n) \leq \Big[\frac{(K+2)D(x_{n-1}, x_n) + (K+1)D(x_n, y_n)}{(K+2)D(x_{n-1}, x_n) + (K+1)D(x_n, y_n) + \alpha}\Big].D(x_{n-1}, x_n).$$

Set

$$c_n = \frac{(K+2)D(x_{n-1}, x_n) + (K+1)D(x_n, y_n)}{(K+2)D(x_{n-1}, x_n) + (K+1)D(x_n, y_n) + \alpha}.$$

Then $0 < c_n < 1$ and

$$D_n < h_n c_n D_{n-1}, \text{ where } D_n = D(x_n, x_{n+1}), D_{n-1} = D(x_{n-1}, x_n).$$

By using an argument similar to that of the proof of Theorem 3.1, we have completes the proof. □

Remark 3.1. If T is a single map, then Theorem 3.1 reduces to Theorem 2.1 and Theorem 3.3 reduces to Theorem 2.4

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