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# THE $\varphi$-INNER DERIVATIONS AND THE GENERALIZED $\varphi$-DERIVATIONS ON BANACH ALGEBRAS

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**Abstract**

This paper presents some properties of the $\varphi$-derivations and the generalized $\varphi$-derivations from a Banach algebra A into a Banach algebra B. The innerness, one of the most attracting aspect of the derivations on Banach algebras, is also considered. The last sections of this work is discussion about the $\varphi$-homomorphisms from $A$ into $B$ and some continuous properties of the generalized $\varphi$-derivations.

## 1 Introduction

We recall that, a linear mapping $d : A \to A$, where $A$ is a Banach algebra, is called a derivation if $d(ab) = d(a)b + ad(b)$, for all $a, b \in A$. An inner derivation on $A$ is a linear mapping $d : A \to A$ satisfying $d(x) = [a, x] = ax - xa$, for all $x \in A$, for some $a \in A$.

Some other concepts of derivations on Banach algebras have been considered. The (bounded) point derivations of a Banach algebra was presented by I. M. Singer and J. Wermer ([7]): a (bounded) linear function $d_\varphi : U \to U$ is called a (bounded) point derivation (associated with $\varphi$) if $d_\varphi(xy) = d_\varphi(x)\varphi(y) + \varphi(x)d_\varphi(y)$, for all $x, y \in U$, where $U$ is a commutative Banach algebra, $\varphi : U \to U$ is a multiplicative linear function. Successionally, the denotation $d_\varphi$ is used for results in this paper.

**Key words:** Banach algebra, derivation, inner derivation, generalized derivation.
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Suppose $\sigma : A \to A$ is a linear mapping. The derivations from a Banach algebra $A$ into an $A$-module $M$ have been studied. The generalized derivations on $M$ is constructed by Gh. Abbaspour, M. S. Moslehian and A. Niknam ([1]) or B. Havala ([4]). The generalized inner derivations on $A$ can be seen in [4] and [5]. Based on the concept of generalized derivations, A. Hosseini, M. Hassanni and A. Niknam ([3]) presented $\sigma$-derivations, $\sigma$-inner derivations and generalized $\sigma$-derivations from $A$ into $M$, where $\sigma$ is a linear mapping on $A$. Moreover, in [3], some results of $\sigma$-algebraic maps, continuity of generalized $\sigma$-derivations were considered. The generalized Jordan left derivations, generalized $(\sigma, \tau)$-Jordan derivations from $A$ into $M$ were constructed by B. Arslan and H. Inceboz ([2])... Our work in this paper is a view on these above concepts, but for the derivations from a Banach algebra into another Banach algebra.

As is well-known, the derivations on Banach algebras are useful in studying curvature and torsion of these algebra. In 2010, the some properties of Lie derivations of forms on the Banach commutative algebra $B$ was applied for studying curvature and torsion of $B$ by A. Ya. Sultanov ([8]). In [6], N. H. Quang, K. P. Chi and B. C. Van presented the Lie derivations of the currents on Riemann manifolds and some applications on Lie groups. In the recent years, there are two significant problems of derivations on Banach algebras (in the effort to answer some questions): The first one is innerness, with what conditions a derivation can become an inner derivation? And the second one is continuity, what conditions make a derivation becomes a continuous derivation?

In this paper, we present the definitions and the some properties of the $\varphi$-derivations, the $\varphi$-inner derivations and the generalized $\varphi$-derivations from $A$ into $B$, where $A$ and $B$ are Banach algebras. We also show some results of $\varphi$-homomorphisms and give some continuous conditions of the generalized $\varphi$-derivations

## 2 The $\varphi$-derivations and the $\varphi$-inner derivations

From now on, we make the following assumptions: $A$ and $B$ are Banach algebras on $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$), $\varphi : A \to B$ is a nonzero homomorphism.

**Definition 2.1.** *Let $d_\varphi : A \to B$ be a linear mapping.*
*i) The mapping $d_\varphi$ is called a $\varphi$-derivation from $A$ into $B$ if*

$$d_\varphi(xy) = d_\varphi(x)\varphi(y) + \varphi(x)d_\varphi(y),$$

*for all $x, y \in A$.*

*ii) The mapping $d_\varphi$ is called a $\varphi$-Lie derivation from $A$ into $B$ if*

$$d_\varphi[x, y] = [d_\varphi(x), \varphi(y)] + [\varphi(x), d_\varphi(y)],$$

*for all $x, y \in A$.*

**Example 2.2.** Let $A = \mathbb{R}[x]$ be the algebra of polynomials in one variable on $\mathbb{R}$ and $B = \mathbb{R}$. With $\lambda \in \mathbb{R}$, we defined $\varphi_\lambda : A \to B$ by $f \mapsto \varphi_\lambda(f) = f(\lambda)$ and $d_{\varphi_\lambda} : A \to B$ by $f \mapsto d_{\varphi_\lambda}(f) = f'(\lambda)$ (where $f'(x) = \dfrac{d}{dx}f(x)$). Then, $\varphi_\lambda$ is a homomorphism, $d_{\varphi_\lambda}$ is a $\varphi_\lambda$-derivation and is also a $\varphi_\lambda$-Lie derivation.

It is easy to see that if $A = B$ and $\varphi = id_A$, then the concepts derivation (on $A$) and $\varphi$-derivation (on $A$) are similar. We can also easily check that a $\varphi$-derivation (from $A$ into $B$) is a $\varphi$-Lie derivation.

Now suppose $b \in B$, the mapping $d_b : A \to B$ defined by

$$d_b(x) = [b, \varphi(x)] = b\varphi(x) - \varphi(x)b,$$

for all $x \in A$, is a $\varphi$-derivation.

**Definition 2.3.** *The $\varphi$-derivation $d_b$ is called a $\varphi$-inner derivation corresponds with $b \in B$.*

Since $\varphi$ is continuous, it follows that $d_b$ is continuous.

**Proposition 2.4.** *Let $d_b : A \to B$ be a $\varphi$-inner derivation. Then,*
*i) $d_{b_1+b_2} = d_{b_1} + d_{b_2}$, for all $b_1, b_2 \in B$,*
*ii) $d_{\lambda b} = \lambda d_b$, for all $b \in B$, $\lambda \in \mathbb{K}$,*
*iii) If $c \in G(B)$, then $cd_b = d_{cb}$, for all $b \in B$, where $G(B)$ is the center of $B$ and $(cd_b)(x) = c.d_b(x)$, for all $x \in A$.*

*Proof.* We prove $i)$ and $iii)$.
$i)$ For all $b_1, b_2 \in B$ and $x \in A$,

$$\begin{aligned} d_{b_1+b_2}(x) &= [b_1 + b_2, \varphi(x)] \\ &= (b_1 + b_2)\varphi(x) - \varphi(x)(b_1 + b_2) \\ &= (b_1\varphi(x) - \varphi(x)b_1) + (b_2\varphi(x) - \varphi(x)b_2) \\ &= [b_1, \varphi(x)] + [b_2, \varphi(x)] \\ &= d_{b_1}(x) + d_{b_2}(x). \end{aligned}$$

$iii)$ For all $c \in G(B)$, $b \in B$ and $x \in A$,

$$\begin{aligned} cd_b(x) &= c[b, \varphi(x)] \\ &= c(b\varphi(x) - \varphi(x)b) \\ &= cb\varphi(x) - c\varphi(x)b \\ &= cb\varphi(x) - \varphi(x)cb \\ &= [cb, \varphi(x)] \\ &= d_{cb}(x). \end{aligned}$$

□ Now, we denote by $D(B)$, $DI(B)$, $D_\varphi(A, B)$ and $DI_\varphi(A, B)$ the set of all derivations on $B$, the set of all inner derivations on $B$, the set of all

$\varphi$-derivations from $A$ into $B$ and the set of all $\varphi$-inner derivations from $A$ into $B$, respectively. Then, $D(B)$, $DI(B)$, $D_\varphi(A,B)$ and $DI_\varphi(A,B)$ are modules on $\mathbb{K}$.

Suppose that $f : A \to B$ is a linear mapping and $a \in A$. The Lie derivation of $f$ belongs to $a$ is $L_a f : A \to B$, which is defined by $L_a f(x) = [\varphi(a), f(x)] - f([a,x])$, for all $x \in A$. Let $g : B \to B$ be a linear mapping and $b \in B$, the Lie derivation of $g$ belongs to $b$ is $L_b g : B \to B$, which is defined by $L_b g(y) = [b, g(y)] - g([b,y])$, for all $y \in B$.

Now we consider the innerness of the above Lie derivations.

**Theorem 2.5.**
*i) If $d_\varphi \in D_\varphi(A,B)$, then $L_a d_\varphi \in DI_\varphi(A,B)$, for $a \in A$.*
*ii) If $d \in D(B)$, then $L_b d \in DI(B)$, for $b \in B$.*

*Proof.* $i$) For $a \in A$ and for all $x \in A$,

$$\begin{aligned}
&L_a d_\varphi(x) = [\varphi(a), d_\varphi(x)] - d_\varphi([a,x])\\
&= \varphi(a)d_\varphi(x) - d_\varphi(x)\varphi(a) - d_\varphi(ax) + d_\varphi(xa)\\
&= \varphi(a)d_\varphi(x) - d_\varphi(x)\varphi(a) - d_\varphi(a)\varphi(x) - \varphi(a)d_\varphi(x) + d_\varphi(x)\varphi(a) + \varphi(x)d_\varphi(a)\\
&= -d_\varphi(a)\varphi(x) + \varphi(x)d_\varphi(a)\\
&= [-d_\varphi(a), \varphi(x)]\\
&= d_{(-d_\varphi(a))}(x)\\
&= d_b(x),
\end{aligned}$$

where $b = -d_\varphi(a)$, $a \in A$. Thus, $L_a d_\varphi$ is a $\varphi -$ inner derivation from $A$ into $B$.
$ii$) For $b \in B$ and for all $y \in B$,

$$\begin{aligned}
L_b d(y) &= [b, d(y)] - d([b,y])\\
&= bd(y) - d(y)b - d(by) + d(yb)\\
&= bd(y) - d(y)b - d(b)y - bd(y) + d(y)b + yd(b)\\
&= -d(b)y + yd(b)\\
&= [-d(b), y].
\end{aligned}$$

Thus, $L_b d$ is an inner derivation on $B$. □

We now note the mapping $L_a : D_\varphi(A,B) \to DI_\varphi(A,B)$, $d_\varphi \mapsto L_a d_\varphi$. Since Theorem 2.5, we put the following formulas.

**Corollary 2.6.**
*i) $L_{a_1+a_2} = L_{a_1} + L_{a_2}$, for all $a_1, a_2 \in A$.*
*ii) $L_{\lambda a} = \lambda L_a$, for all $\lambda \in \mathbb{K}$, $a \in A$.*
*iii) If $a_1, a_2 \in A$ and $\varphi(a_2) \in G(B)$, then $L_{a_1 a_2} = \varphi(a_2)L_{a_1} + \varphi(a_1)L_{a_2}$, where $(\varphi(a_2)L_{a_1})(d_\varphi) = \varphi(a_2)(L_{a_1} d_\varphi)$, for all $d_\varphi \in D_\varphi(A,B)$.*
*iv) If $a_1, a_2 \in A$ and $\varphi(a_1), \varphi(a_2) \in G(B)$, then $L_{[a_1,a_2]} = 0$.*

*Proof.* We prove $i)$ and $iii)$.
$i)$ $L_{a_1+a_2}d_\varphi = d_{-d_{\varphi(a_1+a_2)}} = d_{-d_{\varphi(a_1)+\varphi(a_2)}} = d_{-d_{\varphi(a_1)}} + d_{-d_{\varphi(a_2)}} = L_{a_1}d_\varphi + L_{a_2}d_\varphi$, for all $a_1, a_2 \in A$ and for all $d_\varphi \in D_\varphi(A, B)$. Thus, $L_{a_1+a_2} = L_{a_1} + L_{a_2}$.
$iii)$ For $a_1, a_2 \in A$ such that $\varphi(a_2) \in G(B)$,

$$\begin{aligned}
L_{a_1a_2}d_\varphi &= d_{-d_\varphi(a_1a_2)} \\
&= d_{(-d_\varphi(a_1)\varphi(a_2)-\varphi(a_1)d_\varphi(a_2))} \\
&= d_{(-d_\varphi(a_1)\varphi(a_2))} + d_{(-\varphi(a_1)d_\varphi(a_2))} \\
&= d_{(\varphi(a_2)(-d_\varphi(a_1)))} + d_{\varphi(a_1)(-d_\varphi(a_2))} \\
&= \varphi(a_2)d_{-d_\varphi(a_1)} + \varphi(a_1)d_{(-d_\varphi(a_2))} \\
&= \varphi(a_2)L_{a_1}d_\varphi + \varphi(a_1)L_{a_2}d_\varphi \\
&= (\varphi(a_2)L_{a_1} + \varphi(a_1)L_{a_2})(d_\varphi),
\end{aligned}$$

for all $d_\varphi \in D_\varphi(A, B)$. □

We are now going to define the full-back mapping of $d \in D(B)$.

**Definition 2.7.** *Let $d \in D(B)$. The full-back mapping of $d$ is $\varphi^*d : A \to B$ defined by $\varphi^*d(x) = d(\varphi(x))$, for all $x \in A$.*

**Proposition 2.8.** *If $d \in D(B)$ (respectively, $d \in DI(B)$), then $\varphi^*d \in D_\varphi(A, B)$ (respectively, $\varphi^*d \in DI_\varphi(A, B)$).*

*Proof.* For all $d \in D(B)$, $\varphi^*d$ is a linear mapping. Furthermore, for all $x, y \in A$,

$$\begin{aligned}
\varphi^*d(xy) &= d(\varphi(xy)) \\
&= d(\varphi(x)\varphi(y)) \\
&= d(\varphi(x))\varphi(y) + \varphi(x)d(\varphi(y)) \\
&= \varphi^*d(x)\varphi(y) + \varphi(x)\varphi^*d(y).
\end{aligned}$$

Hence, $\varphi^*d$ is a $\varphi-$derivation from $A$ into $B$.

By an argument similar to the previous one we get that if $d \in DI(B)$, then $\varphi^*d \in DI_\varphi(A, B)$. □

Assume $\varphi^* : D(B) \to D_\varphi(A, B)$ is the mapping defined by $\varphi^*(\delta) = \varphi^*\delta$, where $\varphi^*\delta(x) = \delta(\varphi(x))$, for all $x \in A$. Then, $\varphi^*$ is a linear mapping.

Now, for $a \in A$ and $b \in B$, we note the mappings $L_a\varphi^*$ and $\varphi^*L_b$ from $D(B)$ into $D_\varphi(A, B)$, which are defined respectively by $L_a\varphi^*(d) = L_a(\varphi^*d)$ and $\varphi^*L_b(d) = \varphi^*(L_bd)$, for all $d \in D(B)$. Since $\varphi^*d \in D_\varphi(A, B)$ and $L_bd \in DI(B)$, then $L_a(\varphi^*d)$ and $\varphi^*(L_bd)$ are also in $DI_\varphi(A, B)$.

**Proposition 2.9.** $L_a\varphi^* = \varphi^*L_{\varphi(a)}$.

*Proof.* For $a \in A$, for all $d \in D(B)$ and for all $x \in A$,

$$\begin{aligned} L_a(\varphi^* d)(x) &= [\varphi(a), \varphi^* d(x)] - \varphi^* d([a, x]) \\ &= [\varphi(a), d(\varphi(x))] - d(\varphi([a, x])) \\ &= [\varphi(a), d(\varphi(x))] - d(\varphi(ax) - \varphi(xa)) \\ &= [\varphi(a), d(\varphi(x))] - d(\varphi(a)\varphi(x) - \varphi(x)\varphi(a)) \\ &= [\varphi(a), d(\varphi(x))] - d([\varphi(a), \varphi(x)]) \\ &= \varphi^*(L_{\varphi(a)} d)(x). \end{aligned}$$

□

We denote by $D_\varphi(A)$ the set of all $\varphi$-derivations on $A$ and $DI_\varphi(A)$ the set of all inner $\varphi$-derivations on $A$ ($\varphi$ is a homomorphism on $A$).

**Lemma 2.10.** *Let $d_b \in DI_\varphi(A)$. Then, $L_a d_b = d_c$, where $c = [\varphi(a), b]$.*

*Proof.* For all $x \in A$,

$$\begin{aligned} L_a d_b(x) &= [\varphi(a), d_b(x)] - d_b([a, x]) \\ &= (\varphi(a)b - b\varphi(a))\varphi(x) + \varphi(x)(b\varphi(a) - \varphi(a)b) \\ &= c\varphi(x) - \varphi(x)c \\ &= d_c(x), \end{aligned}$$

where $c = [\varphi(a), b]$. □

From Lemma 3.5, we see that the mapping $L_a^2 : D_\varphi(A) \to D_\varphi(A)$ defined by $L_a^2(d_\varphi) = d_p$, where $p = [d_\varphi(a), \varphi(a)]$, is also a $\varphi$-inner derivation on $A$. The following proposition will lead to a result for specification of $\varphi$-inner derivations on $A$.

**Proposition 2.11.** *If $a \in \ker(\varphi)$ or $\varphi(a) \in G(A)$, then $L_a^2 = 0$.*

*Proof.* For all $d_\varphi \in D_\varphi(A)$, $L_a^2(d_\varphi) = d_p$, where $p = [d_\varphi(a), \varphi(a)]$. Since $a \in \ker(\varphi)$ or $\varphi(a) \in G(A)$, $p = 0$. Hence, $d_p = 0$. □

Since $L_a^2 = 0$, there is an exact sequence

$$D_\varphi(A) \xrightarrow{L_a} DI_\varphi(A) \xrightarrow{L_a} 0.$$

This means that if $d_\varphi$ is a $\varphi$-inner derivation, then $d_\varphi$ must be in $\ker(L_a)$ for some $a \in \ker(\varphi)$ or $\varphi(a) \in G(A)$. Hence, we have that

$$DI_\varphi(A) \subset \bigcap_{\substack{a \,\in\, \ker(\varphi) \text{ or} \\ \varphi(a) \,\in\, G(A)}} \ker(L_a).$$

# 3 The generalized $\varphi-$ derivations

Let $M$ be an $A$-module, $\sigma : A \to A$ be a linear mapping and $d : A \to M$ be a $\sigma$-derivation. A linear mapping $\delta : A \to M$ is a *generalized $\sigma$-derivation* if $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)d(b)$, for all $a, b \in A$. The mapping $\delta$ is also called a $(\sigma, d)$-derivation. This concept was presented in [3] and it can also be seen in [2]. Also in [2], the generalized $(\sigma, \tau)$-derivations for a couple $\sigma, \tau \in BL(A)$, where $BL(A)$ is the set of all bounded linear operators acting on $A$, is considered.

In this section, we will construct the generalized $\varphi$-derivations corresponding with $\varphi$-derivations from $A$ into $B$ ($A$ and $B$ are Banach algebras).

**Definition 3.1.** *Let $d_\varphi \in D_\varphi(A, B)$. A linear mapping $\widetilde{d} : A \to B$ is called a generalized $\varphi$-derivation corresponding with $d_\varphi$ if*

$$\widetilde{d}(xy) = \widetilde{d}(x)\varphi(y) + \varphi(x)d_\varphi(y),$$

*for all $x, y \in A$.*

It is easy to see that if $d : A \to B$ is a $\varphi$-derivation, then $d$ is also a generalized $\varphi$-derivation.

**Example 3.2.** Let $a, b \in B$. The mapping $d_{(a,b)} : A \to B$ is defined by

$$d_{(a,b)}(x) = a\varphi(x) + \varphi(x)b,$$

for all $x \in A$. Then, $d_{(a,b)}$ is a generalized $\varphi$-derivation corresponding with $d_{-b}$.

Suppose $A$ and $B$ are unital and $\varphi$ separates elements of $B$. It follows from the separation of $\varphi$ for $B$ that $\varphi(1) = 1$.

**Theorem 3.3.** *Let $d_\varphi \in D_\varphi(A, B)$, $\widetilde{d} : A \to B$ be a generalized $\varphi$-derivation corresponding with $d_\varphi$ and $\widetilde{d}(1) = 1$. Then, $L_a\widetilde{d} \in DI_\varphi(A, B)$, for $a \in A$.*

*Proof.* For all $x \in A$,

$$\begin{aligned}\widetilde{d}(x) = \widetilde{d}(1x) &= \widetilde{d}(1)\varphi(x) + \varphi(1)d_\varphi(x)\\ &= 1\varphi(x) + 1d_\varphi(x)\\ &= (\varphi + d_\varphi)(x).\end{aligned}$$

Hence, $\widetilde{d} = \varphi + d_\varphi$. On the other hand, for $a \in A$ and for all $x \in A$, we have

$$\begin{aligned}
L_a\widetilde{d}(x) &= [\varphi(a), \widetilde{d}(x)] - \widetilde{d}[a, x] \\
&= [\varphi(a), (\varphi + d_\varphi)(x)] - (\varphi + d_\varphi)[a, x] \\
&= [\varphi(a), \varphi(x) + d_\varphi(x)] - \varphi[a, x] - d_\varphi[a, x] \\
&= \varphi(a)(\varphi(x) + d_\varphi(x)) - (\varphi(x) + d_\varphi(x))\varphi(a) - \varphi(ax - xa) - d_\varphi(ax - xa) \\
&= \varphi(a)\varphi(x) + \varphi(a)d_\varphi(x) - \varphi(x)\varphi(a) - d_\varphi(x)\varphi(a) - \varphi(a)\varphi(x) + \varphi(x)\varphi(a) \\
&\quad - d_\varphi(a)\varphi(x) - \varphi(a)d_\varphi(x) + d_\varphi(x)\varphi(a) + \varphi(x)d_\varphi(a) \\
&= -d_\varphi(a)\varphi(x) + \varphi(x)d_\varphi(a) \\
&= b\varphi(x) - \varphi(x)b; \text{ for } b = -d_\varphi(a) \\
&= [b, \varphi(x)] \\
&= d_b(x).
\end{aligned}$$

Hence, $L_a\widetilde{d} \in DI_\varphi(A, B)$. □

For convenience, from now on, a generalized $\varphi$-derivation (associates with $\varphi$-derivation $d$) is called $(\varphi, d)$-derivation. In general, the $\varphi$-derivation $d$ is not unique and it may happen that the continuous property of $d$ and $\widetilde{d}$ is not equivalent. In [3], there are results about the continuous property of $\widetilde{d}$. The following results are considered from the some results in [3], but the works are done on generalized $\varphi$-derivations from a Banach algebra $A$ into a Banach algebra $B$. The statements and proofs of these results in this paper are similar to those in [3].

**Definition 3.4.** *A linear mapping $T : A \to B$ is called a $\varphi$-homomorphism if $T(xy) = T(x)\varphi(y)$, for all $x, y \in A$.*

Clearly, if $B$ is unital and $T$ is a $\varphi$-homomorphism, then $\ker(\varphi) \subseteq \ker(T)$.

**Lemma 3.5.** *Suppose $\varphi$ is a surjective map, $A$ is unital and $T(1)$ is invertible. Then, $T(1)B = T(1)^{-1}B = B$.*

We recall that the separating space $S(T)$ of a linear mapping $T : A \to B$ is the set of all $y \in B$ such that there exists a sequence $\{x_n\} \subseteq A$, $x_n \to 0$ and $T(x_n) \to y$. By the Close graph theorem, when $A$ and $B$ are Banach spaces we have that $T$ is continuous if and only if $S(T) = 0$.

**Proposition 3.6.** *Let $T : A \to B$ be a $\varphi$-homomorphism. Then,*
*i) $S(T)\varphi(A) \subseteq S(T)$.*
*ii) $T(A)S(\varphi) \subseteq S(T)$.*
*iii) Suppose $A$ is unital and $T(1)$ is invertible. Then, $S(T) = T(1)S(\varphi)$. Furthermore, $T$ is surjective if and only if $\varphi$ is surjective, $S(T) = B$ if and only if $S(\varphi) = B$.*
*iv) Suppose $\varphi$ is surjective. Then, $T(A)$ is a right ideal of $B$. Furthermore, $T(A)$ is generated by $T(1)$ if $A$ is unital.*

If $M$ is a subset of an algebra $A$, then the right annihilator $ran(M)$ (respectively, the left annihilator $lan(M)$) of $M$ is the set of all $x \in A$ such that $Mx = 0$ (respectively, $xM = 0$). The annihilator $ann(M)$ of $M$ is the set $ran(M) \cap lan(M)$.

**Corollary 3.7.** *Suppose $T : A \to B$ is a $\varphi$-homomorphism.*
*i) If $\varphi$ is surjective, then $S(T)$ is a right ideal of $B$.*
*ii) If $T$ is continuous and $ran(T(A)) = 0$, then $\varphi$ is continuous.*
*iii) Assume that $A$ is unital and $T(1)$ is invertible. Then, $T$ is continuous if and only if $\varphi$ is continuous.*
*iv) If $A$ is unital, $T(1) = 1$ and $\varphi$ is surjective, then $B = (1)$.*

In [3], the Cohen's factorization property of a Banach algebra was defined: a Banach algebra $A$ has the Cohen's factorization property if for all sequences $\{x_n\} \subset A$ such that $x_n \to 0$ there exists an element $a \in A$ and a sequence $\{y_n\} \subset A$ such that $y_n \to 0$ and $x_n = ay_n$, for all positive integer $n$. When $A$ has the Cohen's factorization property, we have a result on continuous property of the $\varphi$-homomorphisms from $A$ into $B$ as following.

**Proposition 3.8.** *Let $A$ be a Banach algebra within the Cohen's factorization property, $T : A \to B$ be a non-zero $\varphi$-homomorphism and $ran(T(A)) = 0$. Then, $T$ is continuous if and only if $\varphi$ is continuous.*

Suppose $T : A \to B$ is a $\varphi$-homomorphism and $d : A \to B$ is a $\varphi$-derivation. Then, $\widetilde{d} = d+T$ is a linear mapping and $\widetilde{d}(xy) = (d+T)(xy) = (d+T)(x)\varphi(y)+\varphi(x)d(y) = \widetilde{d}(x)\varphi(y) + \varphi(x)d(y)$, for all $x, y \in A$. So $\widetilde{d}$ is a $(\varphi, d)$-derivation.

**Proposition 3.9.** *A linear mapping $\widetilde{d} : A \to B$ is a $(\varphi, d)$-derivation if and only if there exists a $\varphi$-derivation $d : A \to B$ and a $\varphi$-homomorphism $T : A \to B$ such that $\widetilde{d} = d + T$.*

We recall that a Banach algebra $A$ has no zero divisor if for all $x, y \in A$ such that $xy = 0$, then $x = 0$ or $y = 0$.

**Theorem 3.10.**
*i) Suppose $\varphi$ is continuous, $A$ has the Cohen's factorization property and $\widetilde{d} : A \to B$ is a $(\varphi, d)$-derivation. Then, $\widetilde{d}$ is continuous if and only if $d$ is continuous.*
*ii) Suppose $B$ has no zero divisor, $A$ has the Cohen's factorization property and $\widetilde{d} : A \to B$ is a $(\varphi, d)$-derivation, $d \neq 0$. Then, $d$ is continuous if and only if $\widetilde{d}$ is continuous.*
*iii) Suppose $A$ has the Cohen's factorization property and is unital. Suppose $\widetilde{d} : A \to B$ is a $(\varphi, d)$-derivation. If there exists an element $a \in A$ such that $d(a)$ is invertible, then $\widetilde{d}$ is continuous if and only if $d$ is continuous.*

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