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A NOTE ON POSITIVSTELLENSÄTZE FOR MATRIX POLYNOMIALS

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Abstract

In this paper we give a note on the relation of the positivity of polynomial matrices and their homogenizations on basic closed semi-algebraic sets. Base on this relation, we extend Putinar-Vasilescu's Positivstellensatz, in particular, Reznick's Positivstellensatz, and Dickinson-Povh's Positivstellensatz to (not necessarily homogeneous) polynomial matrices. This is a continuation of the work [C.-T. Lê, *Some Positivstellensätze for polynomial matrices*, Positivity **19** (3) (2015), 513-528].

1 Introduction

For a subset G of the ring $\mathbb{R}[X] := \mathbb{R}[X_1, \cdots, X_n]$ of polynomials in n variables $X_1, \cdots, X_n$ with real coefficients, the set

$$K_G := \{x \in \mathbb{R}^n | g(x) \geq 0 \text{ for all } g \in G\}$$

is called a *basic closed semi-algebraic set* in $\mathbb{R}^n$.

A Positivstellensatz basically represents polynomials that are positive (or non-negative) on basic closed semi-algebraic sets. Positivstellensätze have many important applications, e.g. to solve the moment problems and the polynomial optimization problems. Shor ([19], 1987) introduced the idea of applying a convex optimization technique to minimize an unconstrained multivariate polynomial. Nesterov ([12], 2000) exploited the duality of moment cones and cones of non-negative polynomials in a convex optimization framework. A

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milestone in minimizing multivariate polynomials was given by Lasserre ([9], 2001), who realized to apply recent real algebraic results by Putinar [14] to construct a sequence of semidefinite program relaxations whose optima converge to the optimum of a polynomial optimization problem. For more details about applications of Positivstellensätze in these areas, the reader may find in the survey of M. Laurent and references therein ([10], 2009).

Pólya's Positivstellensatz [13] represents polynomials which are positive on the non-negative orthant $\mathbb{R}^n_+$. Putinar and Vasilescu ([15], 1999) described polynomials that are positive on basic closed semi-algebraic sets in $\mathbb{R}^n \setminus \{0\}$. In particular, when the closed semi-algebraic set is the whole space $\mathbb{R}^n$, they obtained again Reznick's Positivstellensatz. Recently, P. Dickinson and J. Povh ([4], 2015) obtained a Postivstellensatz which can be seen as a combination of Pólya's and Putinar-Vasilescu's Positivstellensätze, and gave some applications of their results.

Positivstellensätze for polynomial matrices, i.e. matrices with entries belong to the polynomial ring $\mathbb{R}[X]$, have attracted the interest of many people in recent years. We can tell here the works of V.A. Jakubovi ([7], 1970) represented a univariable polynomial matrix which is positive definite on $\mathbb{R}$ as a sum of two hermitian squares; D. Gondard and P. Ribenboim ([5], 1974) gave a matrix version for Artin's theorem on the Hilbert's seventeenth problem; C.W.J. Hol and C.W. Scherer ([6], 2006), among others, extended Polya's and Putinar's Positivstellensätze for polynomial matrices; K. Schmüdgen ([18], 2009) introduced some basic concepts and first ideas for noncommutative real algebraic geometry; J. Cimpri ([2], 2012) introduced also real algebraic geometry for matrices over commutative rings, in which, he proved a matrix version for Krivine-Stengle's Positivstellensatz; I. Klep and M. Schweighofer ([8], 2010) represented polynomial matrices that are positive definite (resp. not negative semidefinite) on basic closed semi-algebraic sets whose corresponding quadratic modules are Archimedean, obtained again Putinar's Positivstellensatz for polynomial matrices; J. Cimpri and A. Zalar ([3], 2013) studied moment problems for operator polynomials, in particular, they obtained a matrix version of Schmüdgen's Positivstellensatz,... In particular, the work of C.-T. Lê ([11], 2015) introduced matrix versions for Positivstellensätze of Krivine-Stengle, Schweighofer, Scheiderer's local-global principle, Scheiderer's Hessian criterion and Marshall's boundary Hessian conditions.

The main aim of this paper is to extend Putinar-Vasilescu's and Dickinson-Povh's Positivstellensätze to polynomial matrices. The paper is organized as follows. In section 2 we introduce some basic concepts and results in Real algebraic geometry for matrices. Moreover, we also give in this section a relation between the positivity of polynomial matrices and their homogenizations on the corresponding basic closed semi-algebraic sets. The main results of the paper is presented in Section 3, in which we give a matrix version for Putinar-Vasilescu's Positivstellensatz and Dickinson-Povh's Positivstellensatz.

2 Preliminaries

2.1 Some basic concepts and results in Real algebraic geometry for matrices

In this section we shall recall some basis concepts and facts in Real algebraic geometry for matrices over commutative rings which are proposed by Schmüdgen ([16], [17], [18]) and Cimpri ([1], [2]). See also in [11].

For $t \in \mathbb{N}^*$, let $\mathcal{M}_t(R)$ denote the ring of $t \times t$ matrices with entries from a commutative unital ring R. Denote by $\mathcal{S}_t(R)$ the subring of $\mathcal{M}_t(R)$ consisting of all symmetric matrices. A subset $\mathcal{M}$ of $\mathcal{S}_t(R)$ is called a *quadratic module* if

$$\mathbf{I}_t \in \mathcal{M}, \quad \mathcal{M} + \mathcal{M} \subseteq \mathcal{M}, \quad \mathbf{A}^T \mathcal{M} \mathbf{A} \subseteq \mathcal{M}, \forall \mathbf{A} \in \mathcal{M}_t(R).$$

The smallest quadratic module which contains a given subset $\mathcal{G}$ of $\mathcal{S}_t(R)$ will be denoted by $\mathcal{M}_{\mathcal{G}}$. It is clear that

$$\mathcal{M}_{\mathcal{G}} = \{\sum_{i,j} \mathbf{A}_{ij}^T \mathbf{G}_i \mathbf{A}_{ij} | \mathbf{G}_i \in \mathcal{G} \cup \{\mathbf{I}_n\}, \mathbf{A}_{ij} \in \mathcal{M}_t(R)\}.$$

In particular, a subset $M \subseteq R$ is a quadratic module if $1 \in M, M + M \subseteq M$, and $a^2 M \subseteq M$ for all $a \in R$. The smallest quadratic module of R which contains a given subset $G \subseteq R$ will be denoted by M_G, and it consists of all finite sums of the form $\sum_{i,j} a_{ij}^2 g_i$, $g_i \in G, a_{ij} \in R$.

A subset $\mathcal{T}$ of $\mathcal{S}_t(R)$ is called a *preordering* if $\mathcal{T}$ is a quadratic module in $\mathcal{M}_t(R)$ and the set $\mathcal{T} \cap (R \cdot \mathbb{I}_t)$ is closed under multiplication. The smallest preordering which contains a given subset $\mathcal{G}$ of $\mathcal{S}_t(R)$ will be denoted by $\mathcal{T}_{\mathcal{G}}$.

For every subset $\mathcal{G}$ of $\mathcal{S}_t(R)$,

$$\mathcal{T}_{\mathcal{G}} = \mathcal{M}_{\mathcal{G} \cup (\prod \mathcal{G}' \cdot \mathbb{I}_t)},$$

where $\prod \mathcal{G}'$ is the set of all finite product of elements from the set $\mathcal{G}' := \{\mathbf{v}^T \mathbf{G} \mathbf{v} | \mathbf{G} \in \mathcal{G}, \mathbf{v} \in R^t\}$.

In particular, a subset $T \subseteq R$ is a preordering if $T + T \subseteq T, T \cdot T \subseteq T, a^2 \in T$ for every $a \in R$. The smallest preordering of R which contains a given subset $G \subseteq R$ will be denoted by T_G. It is clear that

$$T_G = \{ \sum_{\sigma = (\sigma_1, \cdots, \sigma_m) \in \{0,1\}^m} s_\sigma g_1^{\sigma_1} \cdots g_m^{\sigma_m} | m \in \mathbb{N}, g_i \in G, s_\sigma \in \sum R^2 \},$$

where $\sum R^2$ is the set of all sums of squares of finite elements from R.

In the case $\mathcal{G} = \emptyset$, $\sum_t R := \mathcal{M}_\emptyset = \mathcal{T}_\emptyset$ is the set of all finite sums of elements of the form $\mathbf{A}^T \mathbf{A}$, where $\mathbf{A} \in \mathcal{M}_t(R)$, and which is the smallest quadratic module in $\mathcal{M}_t(R)$.

For a quadratic module M in R, denote

$$M^t := \{\sum_i m_i \mathbf{A}_i^T \mathbf{A}_i | m_i \in M, \mathbf{A}_i \in \mathcal{M}_t(R)\}.$$

Then M^t is the smallest quadratic module in $\mathcal{M}_t(R)$ whose intersection with $R \cdot \mathbf{I}_t$ is equal to $M \cdot \mathbf{I}_t$ ([2, Proposition 3]).

In the following we consider R to be the ring $\mathbb{R}[X] := \mathbb{R}[X_1, \cdots, X_n]$ of polynomials in n variables $X_1, \cdots, X_n$ with real coefficients. Then each element $\mathbf{A} \in \mathcal{M}_t(\mathbb{R}[X])$ is a matrix whose entries are polynomials from $\mathbb{R}[X]$, called a *polynomial matrix*. Each element $\mathbf{A} \in \mathcal{M}_t(\mathbb{R}[X])$ is also called a *matrix polynomial*, because it can be viewed as a polynomial in $X_1, \cdots, X_n$ whose entries from $\mathcal{M}_t(\mathbb{R})$. Namely, we can write $\mathbf{A}$ as

$$\mathbf{A} = \sum_{|\alpha|=0}^{d} \mathbf{A}_\alpha X^\alpha,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, $\mathbf{A}_\alpha \in \mathcal{M}_t(\mathbb{R})$, d is the maximum over all degree of entries of $\mathbf{A}$. To unify notation, throughout the paper each element of $\mathcal{M}_t(\mathbb{R}[X])$ is called a *polynomial matrix*.

For any polynomial matrix $\mathbf{A} \in \mathcal{M}_t(\mathbb{R}[X])$ and for any subset $K \subseteq \mathbb{R}^n$, by $\mathbf{A} \succcurlyeq \mathbf{0}$ on K means that for any $x \in K$, the matrix $\mathbf{A}(x)$ is *positive semidefinite*, i.e. for every $\mathbf{v} \in \mathbb{R}^t, \mathbf{v}^T \mathbf{A}(x) \mathbf{v} \geq 0$. Similarly, by $\mathbf{A} \succ \mathbf{0}$ on K means that for any $x \in K$, the matrix $\mathbf{A}(x)$ is *positive definite*, i.e. for every $\mathbf{v} \in \mathbb{R}^t \setminus \{0\}$, $\mathbf{v}^T \mathbf{A}(x) \mathbf{v} > 0$.

We associate to every subset $\mathcal{G} \subseteq \mathcal{S}_t(\mathbb{R}[X])$ the basic closed semi-algebraic set

$$\mathbf{K}_\mathcal{G} := \{x \in \mathbb{R}^n | \mathbf{G}(x) \succcurlyeq \mathbf{0}, \forall \mathbf{G} \in \mathcal{G}\}.$$

In particular, for a subset G of $\mathbb{R}[X]$,

$$K_G = \{x \in \mathbb{R}^n | g(x) \geq 0, \forall g \in G\}.$$

The following result of Cimpri ([2]) shows that the set $\mathbf{K}_\mathcal{G}$ can be determined by *scalars*, i.e. by polynomials in $\mathbb{R}[X]$.

Lemma 2.1.1 ([2, Proposition 5]). *Let $\mathcal{G} \subseteq \mathcal{S}_t(\mathbb{R}[X])$. Then there exists a subset G of $\mathbb{R}[X]$ with the following properties:*

(1) $K_G = \mathbf{K}_\mathcal{G}$;

(2) $(M_G)^t \subseteq \mathcal{M}_\mathcal{G}$;

(3) $(T_G)^t \subseteq \mathcal{T}_\mathcal{G}$.

Moreover, if $\mathcal{G}$ is finite then G can be chosen to be finite.

It is well-known that every symmetric scalar matrix $\mathbf{A} \in \mathcal{S}_t(\mathbb{R})$ can be diagonalized by an orthogonal matrix $\mathbf{O} \in \mathcal{M}_t(\mathbb{R})$. For a polynomial matrix $\mathbf{A}$ in $\mathcal{S}_t(\mathbb{R}[X])$, it is in general no longer true, because the matrix $\mathbf{O}$ may have rational entries (quotients of two polynomials in $\mathbb{R}[X]$). However, Schmüdgen ([18], 2009) showed that every symmetric polynomial matrix can be diagonalized by an invertible matrix in $\mathcal{M}_t(\mathbb{R}[X])$ with a quotient by a non-zero polynomial in $\mathbb{R}[X]$. Moreover, in some special cases (e.g. that symmetric polynomial is in *standard form*), that invertible matrix can be chosen to be lower triangular.

Lemma 2.1.2 ([18, Corollary 9]). *Let $\mathbf{A} \in \mathcal{S}_t(\mathbb{R}[X])$. Then there exist non-zero polynomials $b, d_j \in \mathbb{R}[X]$, $j = 1, \cdots, r$, $r \leq n$, and matrices $\mathbf{X}_+, \mathbf{X}_- \in \mathcal{M}_t(\mathbb{R}[X])$ such that*

$$\mathbf{X}_+\mathbf{X}_- = \mathbf{X}_-\mathbf{X}_+ = b\mathbf{I}_t, \quad b^2\mathbf{A} = \mathbf{X}_+\mathbf{D}\mathbf{X}_+^T, \quad \mathbf{D} = \mathbf{X}_-\mathbf{A}\mathbf{X}_-^T,$$

where $\mathbf{D}$ is the $t \times t$ diagonal matrix $D(d_1, \cdots, d_r)$.

Remark 2.1.3. With $\mathbf{A}, \mathbf{D}$ as above, for any subset $K \subseteq \mathbb{R}^n$, if $\mathbf{A} \succ \mathbf{0}$ (resp. $\mathbf{A} \succcurlyeq \mathbf{0}$) on K then $\mathbf{D} \succ \mathbf{0}$ (resp. $\mathbf{D} \succcurlyeq \mathbf{0}$) on K.

2.2 Positivity of polynomial matrices and their homogenizations

A polynomial $f \in \mathbb{R}[X]$ is said to be *homogeneous* of degree d if

$$f(\lambda X_1, ..., \lambda X_n) = \lambda^d f(X_1, ..., X_n),$$

for every $\lambda \neq 0$. Note that each nonzero polynomial $f \in \mathbb{R}[X]$ of degree d can be decomposed, in a unique way, as

$$f = f_0 + f_1 + \cdots + f_d,$$

where f_i denotes the homogeneous component of degree i of f, for $i = 1, ..., d$. Moreover, given a polynomial $f \in \mathbb{R}[X]$ of degree d, we can get a homogeneous polynomial $\tilde{f} \in \mathbb{R}[X_0, X_1, \cdots, X_n]$ of degree d by the following way:

$$\tilde{f}(X_0, X_1, ..., X_n) := \begin{cases} X_0^d f\left(\frac{X_1}{X_0}, ..., \frac{X_n}{X_0}\right) & \text{if } X_0 \neq 0 \\ f_d(X_1, ..., X_n) & \text{if } X_0 = 0, \end{cases}$$

where X_0 is a new variable. The polynomial $\tilde{f}$ is called the *homogenization* of f. It is easy to see that

$$\tilde{f}(1, X_1, ..., X_n) = f(X_1, ..., X_n) \text{ and } \tilde{f}(0, X_1, ..., X_n) = f_d(X_1, \cdots, X_n).$$

For any polynomial matrix $\mathbf{F} = (f_{ij}) \in \mathcal{M}_t(\mathbb{R}[X])$ which is of degree d, denote by $\mathbf{F}_d = ((f_{ij})_d)$ the *homogeneous part of degree* d of $\mathbf{F}$. The polynomial matrix $\tilde{F} = (\tilde{f}_{ij})$ is called the *homogenization* of the matrix polynomial $\mathbf{F}$.

The following result gives a relation of positivity of f and $\tilde{f}$ on the coresponding basic closed semi-algebraic sets.

Lemma 2.2.1. *Let* $G = \{g_1, \cdots, g_m\} \subseteq \mathbb{R}[X_1, \cdots, X_n]$ *and* $f \in \mathbb{R}[X_1, \cdots, X_n]$. *Let* $\tilde{f}, \tilde{g}_1, \cdots, \tilde{g}_m \in \mathbb{R}[X_0, \cdots, X_n]$ *be the homogenization of the polynomials* $f, g_1, \cdots, g_m \in \mathbb{R}[X_1, \cdots, X_n]$, *respectively, with* $\deg(f) = 2d, \deg(g_i) = 2d_i, \forall i = 1, \cdots, m$. *Denote* $d' := \max\{d_i, i = 1, \cdots, m\}$, $\tilde{G} := \{\tilde{g}_1, \cdots, \tilde{g}_m\}$, *and*

$$(K_G)_{2d'} = \{x \in \mathbb{R}^n | (g_i)_{2d'} \geq 0, \forall i = 1, \cdots m\}.$$

Note that $(g_i)_{2d'} = 0$ *if* $d' > d_i$. *Then* $\tilde{f} > 0$ *on* $K_{\tilde{G}} \setminus \{0\}$ *if and only if* $f > 0$ *on* K_G *and* $f_{2d} > 0$ *on* $(K_G)_{2d'} \setminus \{0\}$.

Proof. Assume that $\tilde{f} > 0$ on $K_{\tilde{G}} \setminus \{0\}$. Then for each $x \in K_G$, we have $(1, x) \in K_{\tilde{G}} \setminus \{0\}$. It follows that $f(x) = \tilde{f}(1, x) > 0$, i.e. $f > 0$ on K_G. Moreover, for each $x \in (K_G)_{2d'} \setminus \{0\}$ we have $(0, x) \in K_{\tilde{G}} \setminus \{0\}$. So $f_{2d}(x) = \tilde{f}(0, x) > 0$, i.e. $f_{2d} > 0$ on $(K_G)_{2d'} \setminus \{0\}$.

Conversely, assume $f > 0$ on K_G and $f_{2d} > 0$ on $(K_G)_{2d'} \setminus \{0\}$. For each $(x_0, x) \in K_{\tilde{G}} \setminus \{0\}$, we have $\tilde{g}_i(x_0, x) \geq 0$ for all $i = 1, \cdots, m$. If $x_0 = 0$, since $(x_0, x) \neq (0, 0)$, we have $x \neq 0$. Then, for every $i = 1, \cdots, m$,

$$(g_i)_{2d'}(x) = \tilde{g}_i(0, x) \geq 0, \text{ i.e. } x \in (K_G)_{2d'} \setminus \{0\}.$$

It follows that $\tilde{f}(0, x) = f_{2d}(x) > 0$. If $x_0 \neq 0$, by definition we have $\tilde{g}_i(x_0, x) = x_0^{2d_i} g_i\left(\frac{x}{x_0}\right) \geq 0$. Since $x_0 \neq 0$ we have $g_i\left(\frac{x}{x_0}\right) \geq 0, \forall\, 1 \leq i \leq m$, i.e. $\dfrac{x}{x_0} \in K_G$. It follows from the hypothesis that $f\left(\frac{x}{x_0}\right) > 0$. Hence

$$\tilde{f}(x_0, x) = x_0^{2d} f\left(\frac{x}{x_0}\right) > 0,$$

i.e. $\tilde{f} > 0$ on $K_{\tilde{G}} \setminus \{0\}$. The proof is complete. □ By the same argument we get the same conclusion for the positivity of the polynomial f and its homogenization on the intersection of a basic closed semi-algebraic set in $\mathbb{R}^n$ with the positive cone

$$\mathbb{R}^n_+ = \{x = (x_1, .., x_n) \in \mathbb{R}^n | x_i \geq 0, i = 1, ..., n\}.$$

Lemma 2.2.2. *With the notation as in Lemma 2.2.1,* $\tilde{f} > 0$ *on* $\mathbb{R}^{n+1}_+ \cap K_{\tilde{G}} \setminus \{0\}$ *if and only if* $f > 0$ *on* $\mathbb{R}^n_+ \cap K_G$ *and* $f_{2d} > 0$ *on* $\mathbb{R}^n_+ \cap (K_G)_{2d'} \setminus \{0\}$.

Now we introduce similar results for polynomial matrices. Let

$$\mathcal{G} = \{\mathbf{G}_1, \cdots, \mathbf{G}_m\} \subseteq \mathcal{S}_t(\mathbb{R}[X]) \text{ and } \mathbf{F} \in \mathcal{S}_t(\mathbb{R}[X]).$$

Assume $\deg(\mathbf{F}) = 2d, \deg(\mathbf{G_i}) = 2d_i, i = 1, ..., m$. Denote $d' := \max\{d_i | i = 1, \cdots, m\}$, and

$$(K_{\mathcal{G}})_{2d'} := \{x \in \mathbb{R}^n | (\mathbf{G_i})_{2d'}(x) \succcurlyeq \mathbf{0}, \forall i = 1, \cdots, m\}.$$

Let $\tilde{\mathbf{F}}, \tilde{\mathbf{G}}_\mathbf{1}, \cdots, \tilde{\mathbf{G}}_\mathbf{m}$ be respectively the homogenization of the polynomial matrices $\mathbf{F}, \mathbf{G}_1, \cdots, \mathbf{G}_m$. Denote $\widetilde{\mathcal{G}} = \{\tilde{\mathbf{G}}_\mathbf{1}, \cdots, \tilde{\mathbf{G}}_\mathbf{m}\}$, and

$$K_{\widetilde{\mathcal{G}}} = \{(x_0, x) \in \mathbb{R}^{n+1} | \tilde{\mathbf{G}}_i(x_0, x) \succcurlyeq \mathbf{0}, \forall i = 1, \cdots, m\}.$$

Then, by the same argument given in the proof of Lemma 2.2.1, we obtain the following results.

Lemma 2.2.3. *$\tilde{\mathbf{F}} \succ \mathbf{0}$ on $K_{\tilde{\mathcal{G}}} \setminus \{0\}$ if and only if $\mathbf{F} \succ \mathbf{0}$ on $K_{\mathcal{G}}$ and $\mathbf{F}_{2d} \succ \mathbf{0}$ on $(K_{\mathcal{G}})_{2d'} \setminus \{0\}$.*

Lemma 2.2.4. *$\tilde{\mathbf{F}} \succ \mathbf{0}$ on $\mathbb{R}^{n+1}_+ \cap K_{\tilde{\mathcal{G}}} \setminus \{0\}$ if and only if $\mathbf{F} \succ \mathbf{0}$ on $\mathbb{R}^n_+ \cap K_{\mathcal{G}}$ and $\mathbf{F}_{2d} \succ \mathbf{0}$ on $\mathbb{R}^n_+ \cap (K_{\mathcal{G}})_{2d'} \setminus \{0\}$.*

3 Putinar-Vasilescu's and Dickinson-Povh's Positivstellensätze for polynomial matrices

3.1 Putinar-Vasilescu's Positivstellensatz for polynomial matrices

Let us first recall Putinar-Vasilescu's Positivstellensatz for homogeneous polynomials.

Theorem 3.1.1 ([15, Theorem 4.2]). *Let $f, g_1, \cdots, g_m \in \mathbb{R}[X]$ be homogeneous polynomials of even degree and assume that $f > 0$ on $K_G \setminus \{0\}$, where $G = \{g_1, \cdots, g_m\}$. Then there exists an integer $r \geq 0$ such that*

$$(X_1^2 + \cdots + X_n^2)^r f \in M_G.$$

As a corollary of this theorem and Lemma 2.2.1, we obtain the following Putinar-Vasilescu's Positivstellensatz for arbitrary polynomials.

Corollary 3.1.2. *Let $G = \{g_1, \cdots, g_m\} \subseteq \mathbb{R}[X]$ and $f \in \mathbb{R}[X]$. Assume $\deg(f) = 2d, \deg(g_i) = 2d_i, i = 1, ..., m$. Denote $d' := \max\{d_i | i = 1, \cdots, m\}$. If $f > 0$ on K_G and $f_{2d} > 0$ on $(K_G)_{2d'} \setminus \{0\}$, then there exists an integer $r \geq 0$ such that*

$$(1 + X_1^2 + \cdots + X_n^2)^r f \in M_G.$$

Proof. By Lemma 2.2.1 we have $\tilde{f} > 0$ on $K_{\tilde{G}} \setminus \{0\}$, where $\tilde{G} = \{\tilde{g}_1, \cdots, \tilde{g}_m\}$. Applying Theorem 3.1.1 for the homogeneous polynomial $\tilde{f} \in \mathbb{R}[X_0, X_1, \cdots, X_n]$, we have

$$(X_0^2 + X_1^2 + \cdots + X_n^2)^r \tilde{f} \in M_{\tilde{G}},$$

for some $r \in \mathbb{N}$. Replacing $X_0 = 1$, observing that

$$\tilde{g}_i(1, X_1, \cdots, X_n) = g_i(X_1, \cdots, X_n) \text{ for all } i = 1, \cdots, m,$$

we obtain the required result. □ Now we give a matrix version of Putinar-Vasilescu's Positivstellensatz.

Theorem 3.1.3. *Let* $\mathcal{G} = \{\mathbf{G}_1, \cdots, \mathbf{G}_m\} \subseteq \mathcal{S}_t(\mathbb{R}[X])$ *and* $\mathbf{F} \in \mathcal{S}_t(\mathbb{R}[X])$. *Suppose* $\deg(\mathbf{F}) = 2d, \deg(\mathbf{G_i}) = 2d_i, i = 1, ..., m$. *Denote* $d' := \max\{d_i | i = 1, \cdots, m\}$. *Assume that* $\mathbf{F} \succ \mathbf{0}$ *on* $K_{\mathcal{G}}$ *and* $\mathbf{F}_{2d} \succ \mathbf{0}$ *on* $(K_{\mathcal{G}})_{2d'} \setminus \{0\}$. *Then there exists a non-negative integer* r, *a finite subset* $G \subseteq \mathbb{R}[X]$ *and*

(i) a matrix $\mathbf{X} \in \mathcal{M}_t(\mathbb{R}[X])$ *such that*
$$(1 + X_1^2 + \cdots + X_n^2)^r \mathbf{X}\mathbf{F}\mathbf{X}^T \in (M_G)^t \subseteq \mathcal{M}_{\mathcal{G}};$$

(ii) a non-zero polynomial $b \in \mathbb{R}[X]$ *such that*
$$b^2(1 + X_1^2 + \cdots + X_n^2)^r \mathbf{F} \in (M_G)^t \subseteq \mathcal{M}_{\mathcal{G}}.$$

Proof. Firstly, we assume that $\mathbf{F} = D(d_1, \cdots, d_r), r \le t$ is a diagonal polynomial matrix. Then $\tilde{\mathbf{F}} = D(\tilde{d}_1, \cdots, \tilde{d}_r)$. It follows from the hypothesis of $\mathbf{F}$ and Lemma 2.2.3 that $\tilde{\mathbf{F}} \succ \mathbf{0}$ on $K_{\tilde{\mathcal{G}}} \setminus \{0\}$, where $\tilde{\mathcal{G}} = \{\tilde{\mathbf{G}}_1, \cdots, \tilde{\mathbf{G}}_m\}$. This implies $r = t$ and $\tilde{d}_i > 0$ on $K_{\tilde{\mathcal{G}}} \setminus \{0\}$ for every $i = 1, \cdots, t$. Then it follows from Lemma 2.1.1 that there exists a finite subset of homogeneous polynomials $\tilde{G} = \{\tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_k\} \subseteq \mathbb{R}[X_0, X]$ such that $K_{\tilde{G}} = K_{\tilde{\mathcal{G}}}, (M_{\tilde{G}})^t \subseteq \mathcal{M}_{\tilde{\mathcal{G}}}$. Put $G = \{g_1, \cdots, g_k\}$, where $g_j(X_1, \cdots, X_n) = \tilde{g}_j(1, X_1, \cdots, X_n)$ for every $j = 1, \cdots, k$. It follows from Theorem 3.1.1 that for each $i = 1, \cdots, t$, there exists an integer $r_i \ge 0$ such that

$$(1 + X_1^2 + \cdots + X_n^2)^{r_i} d_i \in M_G.$$

Let $r = \max\{r_i, i = 1, \cdots, t\}$. Then for every $i = 1, \cdots, t$, we have

$$(1 + X_1^2 + \cdots + X_n^2)^r d_i \in M_G.$$

Therefore $(1 + X_1^2 + \cdots + X_n^2)^r \mathbf{D} \in (M_G)^t$. This implies that

$$(1 + X_1^2 + \cdots + X_n^2)^r \mathbf{D} \in (M_G)^t \subseteq \mathcal{M}_{\mathcal{G}}.$$

In the general case of $\mathbf{F} \in \mathcal{S}_t(\mathbb{R}[X])$, it follows from Lemma 2.1.2 that there exist non-zero polynomials $b, d_j \in \mathbb{R}[X], j = 1, \cdots, r, r \le t$, and matrices $\mathbf{X}_+, \mathbf{X}_- \in \mathcal{M}_t(\mathbb{R}[X])$ such that

$$\mathbf{X}_+\mathbf{X}_- = \mathbf{X}_-\mathbf{X}_+ = bI_t, b^2\mathbf{F} = \mathbf{X}_+\mathbf{D}{\mathbf{X}_+}^T, \mathbf{D} = \mathbf{X}_-\mathbf{F}{\mathbf{X}_-}^T, \tag{3.1}$$

where $\mathbf{D} = \mathbf{D}(d_1, \cdots, d_r)$. By assumption, $\mathbf{F} \succ \mathbf{0}$ on $K_{\mathcal{G}}$, hence by Remark 2.1.3 we have $\mathbf{D} \succ \mathbf{0}$ on $K_{\mathcal{G}}$. Similarly, since $\mathbf{F_{2d}} \succ \mathbf{0}$ on $(K_{\mathcal{G}})_{2d'} \setminus \{0\}$ we have also $\mathbf{D}_m \succ \mathbf{0}$ on $(K_{\mathcal{G}})_{2d'} \setminus \{0\}$ where m is the degree of the polynomial matrix

D. By the first step of the proof, there exists a non-negative integer r such that $(1+X_1^2+\cdots+X_n^2)^r\mathbf{D} \in (M_G)^n \subseteq \mathcal{M}_{\mathcal{G}}$. Then (3.1) yields
(i) $(1+X_1^2+\cdots+X_n^2)^r\mathbf{X}_-\mathbf{F}\mathbf{X}_-{}^T \in (M_G)^t \subseteq \mathcal{M}_{\mathcal{G}}$;
(ii) $b^2(1+X_1^2+\cdots+X_n^2)^r\mathbf{F} = (1+X_1^2+\cdots+X_n^2)^r\mathbf{X}_+\mathbf{D}\mathbf{X}_+^T \in (M_G)^t \subseteq \mathcal{M}_{\mathcal{G}}$.
This completes the proof. $\square$ By setting $\mathcal{G}=\emptyset$, we obtain the following consequence which is a matrix version of Reznick's Positivstellensatz (cf. [15, Corollary 4.3]).

Corollary 3.1.4. *Let $\mathbf{F} \in S_t(\mathbb{R}[X])$ be a symmetric polynomial matrix of degree $2d$. Assume $\mathbf{F} \succ \mathbf{0}$ on $\mathbb{R}^n$ and $\mathbf{F}_{2d} \succ \mathbf{0}$ on $\mathbb{R}^n \setminus \{0\}$. Then there exists a non-negative integer r and*

(i) a polynomial matrix $\mathbf{X} \in \mathcal{M}_t(\mathbb{R}[X])$ such that
$$(1+X_1^2+\cdots+X_n^2)^r\mathbf{X}\mathbf{F}\mathbf{X}^T \in \textstyle\sum_t \mathbb{R}[X];$$

(ii) a non-zero polynomial $b \in \mathbb{R}[X]$ such that
$$b^2(1+X_1^2+\cdots+X_n^2)^r\mathbf{F} \in \textstyle\sum_t \mathbb{R}[X].$$

3.2 Dickinson-Povh's Positivstellensatz for polynomial matrices

The Dickinson-Povh's Positivstellensatz for homogeneous polynomials is stated as follows.

Theorem 3.2.1 ([4, Theorem 3.5]). *Let $f, g_1, \cdots, g_m \in \mathbb{R}[X]$ be homogeneous polynomials of even degree. Denote $G = \{g_1, \cdots, g_m\}$. If $f > 0$ on $\mathbb{R}^n_+ \cap K_G \setminus \{0\}$, then there exist a non-negative integer r and homogeneous polynomials $h_1, \cdots, h_m$ whose coefficients are positive such that*

$$(X_1+\cdots+X_n)^r f = \sum_{i=1}^m g_i h_i.$$

By Lemma 2.2.2, we obtain the following result which is a non-homogeneous version of Dickinson-Povh's Positivstellensatz.

Corollary 3.2.2. *Let $G = \{g_1, \cdots, g_m\} \subseteq \mathbb{R}[X]$ and $f \in \mathbb{R}[X]$. Suppose that $\deg(f) = 2d, \deg(g_i) = 2d_i, \forall i = 1, \cdots, m$. Denote $d' := \max\{d_i | i = 1, \cdots, m\}$. Assume $f > 0$ on $\mathbb{R}^n_+ \cap K_G$ and $f_{2d} > 0$ on $\mathbb{R}^n_+ \cap (K_G)_{2d'} \setminus \{0\}$. Then there exist a non-negative integer r and polynomials $h_1, \cdots, h_m \in \mathbb{R}[X]$ whose coefficients are non-negative such that*

$$(1+X_1+\cdots+X_n)^r f = \sum_{i=1}^m g_i h_i.$$

Proof. It follows from Lemma 2.2.2 that $\tilde{f} > 0$ on $\mathbb{R}_+^{n+1} \cap K_{\tilde{G}} \setminus \{0\}$, where $\tilde{G} = \{\tilde{g}_1, \cdots, \tilde{g}_m\}$. It follows from Theorem 3.2.1 that there exist a non-negative integer r and homogeneous polynomials $\tilde{h}_1, \cdots, \tilde{h}_m$ whose coefficients are positive such that

$$(X_0 + X_1 + \cdots + X_n)^r \tilde{f} = \sum_{i=1}^{m} \tilde{g}_i \tilde{h}_i.$$

Replacing $X_0 = 1$ we get the expected result, with

$$h_i(X_1, \cdots, X_n) := \tilde{h}_i(1, X_1, \cdots, X_n), \text{ for all } i = 1, \cdots, n.$$

□

Now we give a matrix version of Dickinson-Povh's Positivstellensatz.

Theorem 3.2.3. *Let* $\mathcal{G} = \{\mathbf{G}_1, \cdots, \mathbf{G}_m\} \subseteq \mathcal{S}_t(\mathbb{R}[X])$ *and* $\mathbf{F} \in \mathcal{S}_t(\mathbb{R}[X])$. *Suppose* $\deg(\mathbf{F}) = 2d, \deg(\mathbf{G_i}) = 2d_i, i = 1, ..., m$. *Denote* $d' := \max\{d_i | i = 1, \cdots, m\}$. *If* $\mathbf{F} \succ \mathbf{0}$ *on* $\mathbb{R}_+^n \cap \mathbf{K}_{\mathcal{G}}$ *and* $\mathbf{F}_{2d} \succ \mathbf{0}$ *on* $\mathbb{R}_+^n \cap (\mathbf{K}_{\mathcal{G}})_{2d'} \setminus \{0\}$, *then there exist a non-negative integer* r, *a finite subset* $G = \{g_1, \cdots, g_k\} \subseteq \mathbb{R}[X]$ *and*

(i) positive semidefinite matrices $\mathbf{H}_1, \cdots, \mathbf{H}_k \in \mathcal{S}_t(\mathbb{R}[X])$ *and a matrix* $\mathbf{X} \in \mathcal{M}_t(\mathbb{R}[X])$ *such that*

$$(1 + X_1 + \cdots + X_n)^r \mathbf{X}\mathbf{F}\mathbf{X}^T = \sum_{j=1}^{k} \mathbf{H}_j g_j;$$

(ii) positive semidefinite matrices $\mathbf{H'}_1, \cdots, \mathbf{H'}_k \in \mathcal{S}_t(\mathbb{R}[\mathbb{X}])$ *and a non-zero polynomial* $b \in \mathbb{R}[X]$ *such that*

$$b^2(1 + X_1 + \cdots + X_n)^r \mathbf{F} = \sum_{j=1}^{k} \mathbf{H'}_j g_j.$$

Proof. Firstly, we suppose that $\mathbf{F} = \mathbf{D}(d_1, \cdots, d_r), r \leq t, deg(F) = 2d$. Then $\tilde{\mathbf{F}} = \mathbf{D}(\tilde{d}_1, \cdots, \tilde{d}_r)$. It follows from Lemma 2.2.4 that $\tilde{\mathbf{F}} \succ \mathbf{0}$ on $\mathbb{R}_+^{n+1} \cap K_{\tilde{\mathcal{G}}} \setminus \{0\}$. Hence $r = t$ and $\tilde{d}_i > 0$ on $\mathbb{R}_+^{n+1} \cap K_{\tilde{\mathcal{G}}} \setminus \{0\}$, for every $i = 1, \cdots, t$. Now we get from Lemma 2.1.1 a finite subset of homogeneous polynomials $\tilde{G} = \{\tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_k\} \subseteq \mathbb{R}[X_0, X]$ such that $K_{\tilde{G}} = K_{\tilde{\mathcal{G}}}$. It shows that $\tilde{d}_i > 0$ on $\mathbb{R}_+^{n+1} \cap K_{\tilde{G}} \setminus \{0\}$ for every $i = 1, \cdots, t$. Then, by Theorem 3.2.1, for each $i = 1, \cdots, t$, there exist non-negative integers r_i and homogeneous polynomials $\tilde{h}_{i1}, \cdots, \tilde{h}_{ik}$ whose coefficients are non-negative and satisfy

$$(X_0 + X_1 + \cdots + X_n)^{r_i} \tilde{d}_i = \sum_{j=1}^{k} \tilde{h}_{ij} \tilde{g}_j.$$

Let $r = \max\{r_i, i = 1, \cdots, t\}$. Then for every $i = 1, \cdots, t$, we have

$$(X_0 + X_1 + \cdots + X_n)^r \tilde{d}_i = \sum_{j=1}^{k} \tilde{h'}_{ij} \tilde{g}_j$$

for new homogeneous polynomials $\tilde{h'}_{ij}$. Consequently,

$$(X_0 + X_1 + \cdots + X_n)^r \tilde{\mathbf{F}} = \sum_{j=1}^{k} \tilde{\mathbf{H}}_j \tilde{g}_j,$$

where $\tilde{\mathbf{H}}_j = D(\tilde{h'}_{1j}, \tilde{h'}_{2j}, \cdots, \tilde{h'}_{tj}) \in \mathcal{S}_t(\mathbb{R}[X_0, X])$ is a homogeneous polynomial matrix whose coefficients are positive, for every $j = 1, \cdots, k$. Setting $X_0 = 1$, we get

$$(1 + X_1 + \cdots + X_n)^r \mathbf{F} = \sum_{j=1}^{k} \mathbf{H}_j g_j,$$

with $\mathbf{F} = \tilde{\mathbf{F}}(1, X), \mathbf{H}_j = \tilde{\mathbf{H}}_j(1, X), g_j = \tilde{g}_j(1, X)$, for every $j = 1, \cdots, k$.

Now we consider the general case of $\mathbf{F} \in \mathcal{S}_t(\mathbb{R}[X])$. Due to Lemma 2.1.2, there exist non-zero polynomials $b, d_j \in \mathbb{R}[X], j = 1, \cdots, r, r \le t$, and matrices $X_+, X_- \in \mathcal{M}_t(\mathbb{R}[X])$ such that

$$\mathbf{X}_+\mathbf{X}_- = \mathbf{X}_-\mathbf{X}_+ = bI_t, b^2\mathbf{F} = \mathbf{X}_+\mathbf{D}{\mathbf{X}_+}^T, \mathbf{D} = \mathbf{X}_-\mathbf{F}{\mathbf{X}_-}^T, \tag{3.2}$$

where $\mathbf{D} = D(d_1, \cdots, d_r)$. Since $\mathbf{F} \succ \mathbf{0}$ on $\mathbb{R}^n_+ \cap K_{\mathcal{G}}$ and $\mathbf{F}_{2d} \succ \mathbf{0}$ on $\mathbb{R}^n_+ \cap (K_{\mathcal{G}})_{2d} \setminus \{0\}$, this implies that $\mathbf{D} \succ \mathbf{0}$ on $\mathbb{R}^n_+ \cap K_{\mathcal{G}}$ and $\mathbf{D}_s \succ \mathbf{0}$ on $\mathbb{R}^n_+ \cap (K_{\mathcal{G}})_{2d'} \setminus \{0\}$, where $\deg(\mathbf{D}) = s$. Using the obtained result for diagonal polynomial matrices in the first step, there exist an integer $r \ge 0$, positive semidefinite matrices $\mathbf{H}_1, \cdots, \mathbf{H}_k \in \mathbb{R}[X]$ such that

$$(1 + X_1 + \cdots + X_n)^r \mathbf{D} = \sum_{j=1}^{k} \mathbf{H}_j g_j.$$

Using the relations (3.2), we have

(i) $(1 + X_1 + \cdots + X_n)^r \mathbf{X}_-\mathbf{F}{\mathbf{X}_-}^T = (1 + X_1 + \cdots + X_n)^r \mathbf{D} = \sum_{j=1}^{k} \mathbf{H}_j g_j$;

(ii) $$b^2(1 + X_1 + \cdots + X_n)^r \mathbf{F} = \mathbf{X}_+ \left((1 + X_1 + \cdots + X_n)^r \mathbf{D}\right) {\mathbf{X}_+}^T = \mathbf{X}_+ \sum_{j=1}^{k} \mathbf{H}_j g_j {\mathbf{X}_+}^T = \sum_{j=1}^{k} \mathbf{H'}_j g_j,$$

where $\mathbf{H'}_j = \mathbf{X}_+\mathbf{H}_j{\mathbf{X}_+}^T \in \mathcal{M}_t(\mathbb{R}[X])$, for every $j = 1, \cdots, k$. As $\mathbf{H}_j$ is positive semidefinite matrices, so is $\mathbf{H'}_j$. This completes the proof. $\square$

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