A NOTE ON PRIME IDEALS OF IFP-RINGS AND THEIR EXTENSIONS

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Abstract

Let \( R \) be a ring, \( \sigma \) an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \). Let further \( \sigma \) be such that \( a\sigma(a) \in N(R) \) if and only if \( a \in N(R) \) for \( a \in R \), where \( N(R) \) is the set of nilpotent elements of \( R \). We recall that a ring \( R \) is called an IFP-ring if for \( a, b \in R, \) \( ab = 0 \) implies \( aRb = 0 \). In this paper we study the associated prime ideals of Ore extension \( R[x; \sigma, \delta] \) and we prove the following in this direction:

Let \( R \) be a right Noetherian IFP-ring, which is also an algebra over \( \mathbb{Q} \) (\( \mathbb{Q} \) is the field of rational numbers), \( \sigma \) and \( \delta \) as above. Then \( P \) is an associated prime ideal of \( R[x; \sigma, \delta] \) (viewed as a right module over itself) if and only if there exists an associated prime ideal \( U \) of \( R \) such that \( (P \cap R)[x; \sigma, \delta] = P \) and \( P \cap R = U \).

1 Introduction

Notation: We follow the notation and conventions of [3]. All rings are associative with 1. For any subset \( J \) of a right \( R \)-module \( M \), annihilator of \( J \) is denoted by \( Ann(J) \). \( Spec(R) \) denotes the set of prime ideals of \( R \), the set of associated prime ideals of \( R \) (viewed as a right module over itself) is denoted by \( Ass(R_R) \). \( MinSpec(R) \) denotes the set of minimal prime ideals of \( R \). Let \( R \)

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be a right Noetherian ring. For any uniform right \( R \)-module \( J \), the assassinator of \( J \) is denoted by \( \text{Ass}(J) \). Let \( M \) be a right \( R \)-module. Consider the set \( \{ \text{Ass}(J) \mid J \text{ is a uniform right } R\text{-submodule of } M \} \). We denote this set by \( \mathcal{A}(M_R) \).

**Remark 1.1.** If \( R \) is viewed as a right module over itself, we note that \( \text{Ass}(R_R) = \mathcal{A}(R_R) \) (5Y of Goodearl and Warfield [5]).

**Ore Extensions:** Let \( R \) bear in mind an endomorphism of \( R \). Recall that a \( \sigma \)-derivation of \( R \) is an additive map \( \delta : R \rightarrow R \) such that \( \delta(ab) = \delta(a)\sigma(b) + a\delta(b) \), for all \( a, b \in R \). In case \( \sigma \) is the identity map, \( \delta \) is called just a derivation of \( R \).

The Ore extension (or the skew polynomial ring) over \( R \) in an indeterminate \( x \) is:

\[
R[x; \sigma, \delta] = \{ f(x) = x^n a_n + ... + a_0 \mid a_i \in R \}
\]

with \( ax = x\sigma(a) + \delta(a) \) for all \( a \in R \). This definition of non-commutative polynomial rings was first introduced by Ore 1933, who combined earlier ideas of Hilbert (in the case \( \delta = 0 \)) and Schlessinger (in the case \( \sigma = 1 \)). We denote the Ore extension \( R[x; \sigma, \delta] \) by \( O(R) \). An ideal \( I \) of a ring \( R \) is called \( \sigma \)-invariant if \( \sigma(I) = I \) and is called \( \delta \)-invariant if \( \delta(I) \subseteq I \). If an ideal \( I \) of \( R \) is \( \sigma \)-invariant and \( \delta \)-invariant, then \( I[x; \sigma, \delta] \) is an ideal of \( O(R) \) and as usual we denote it by \( O(I) \).

**Definition 1.2.** A ring \( R \) is called 2-primal if and only if \( P(R) = N(R) \), where \( P(R) \) is the prime radical of \( R \) and \( N(R) \) is the set of nilpotent elements of \( R \) (a familiar property of commutative rings). Some of the fundamental properties of 2-primal rings are developed in [6], [12] and [13]. (N. B. The terminology is not uniform: 2-primal rings are called “N-rings” in [6], and, under an equivalent definition, called “weakly symmetric” in [13]).

An ideal \( I \) of a ring \( R \) is called completely semiprime if \( a^2 \in I \) implies \( a \in I \), where \( a \in R \).

**Weak \( \sigma \)-rigid rings and IFP-rings:**

**Definition 1.3.** (Kwak [8]). Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \). Then \( R \) is said to be a \( \sigma(*) \)-ring if \( a\sigma(a) \in P(R) \) implies \( a \in P(R) \) for \( a \in R \).

**Example 1.4.** Let \( R = \mathbb{Z}[\sqrt{-2}] \). Let \( \sigma : R \rightarrow R \) be an endomorphism defined by \( \sigma(a + b\sqrt{-2}) = a - b\sqrt{-2} \). Then \( R \) is a \( \sigma(*) \)-ring.

Ouyang in [10] introduced weak \( \sigma \)-rigid rings, where \( \sigma \) is an endomorphism of ring \( R \). These rings are related to 2-primal rings.

**Definition 1.5.** (Ouyang [10]). Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \) such that \( a\sigma(a) \in N(R) \) if and only if \( a \in N(R) \) for \( a \in R \). Then \( R \) is called a weak \( \sigma \)-rigid ring.
Example 1.6. Assume that $W_1[F]$ is the first Weyl algebra over a field $F$ of characteristic zero. Then $W_1[F] = F[\mu, \lambda]$, the polynomial ring with indeterminates $\mu$ and $\lambda$ with $\lambda \mu = \mu \lambda + 1$. Now let $R$ be the ring \( \begin{pmatrix} W_1[F] & W_1[F] \\ 0 & W_1[F] \end{pmatrix} \). Consider the following element in $R$: \( \begin{pmatrix} \mu \lambda & 0 \\ 0 & 0 \end{pmatrix} \). Now the prime radical $P(R)$ of $R$ is \( \begin{pmatrix} 0 & W_1[F] \\ 0 & 0 \end{pmatrix} \). Define an endomorphism $\sigma : R \to R$ by $\sigma(\begin{pmatrix} \mu \lambda & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}$. Then $R$ is a weak $\sigma$-rigid ring.

Definition 1.7. (Shin [12]). Let $R$ be a ring. Then $R$ is called an IFP-ring (or Ring with Insert Factory Property) if for $a, b \in R$, $ab = 0$ implies $aRb = 0$. Also known as IFP-ring.

Example 1.8. (1) Let $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; a, b \in \mathbb{Z} \right\}$. The only matrices $A$ and $B$ satisfying $AB = 0$ are of the type \( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \); $a, b \in \mathbb{Z}$. i.e., $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$.

Now for all $K = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in R$, $AB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ implies $AKB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$

\[ = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \]

\[ = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & db \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \]

This implies $R$ is an IFP-ring.

(2) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}; a, b \in \mathbb{Z} \right\}$. Then the only matrices $A$ and $B$ satisfying $AB = 0$ are of the type $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$; $a, b \in \mathbb{Z}$.

Now let $a, b, c$ and $d \neq 0$ then for all $K = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in R$
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\[ AB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

But \[ AKB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \]
\[ = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \]
\[ = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & db \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & adb \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

This implies \( R \) is not an IFP-ring.

(3) (Example (5.3) of [12]). Let \( F = \mathbb{Z}_2(y) \) be the field of rational functions over \( \mathbb{Z}_2 \) with \( y \) an indeterminate. Consider the ring \( R = \{ f(x) \in F[x] \mid xy + yx = 1 \} \). Then clearly \( R \) is a domain, so it is reduced and hence an IFP-ring.

Reduced rings (i.e., rings without nonzero nilpotent elements) are obviously IFP-rings, right (left) duo rings are IFP-rings by ([12], Lemma 1.2). Shin showed that IFP-rings are 2-primal in ([12], Theorem 1.5), and so reduced rings are 2-primal.

Lemma 1.9. Let \( R \) be a ring. Let \( \sigma \) be an automorphism of \( R \).
1. If \( P \) is a prime ideal of \( S(R) \) such that \( x \notin P \), then \( P \cap R \) is a prime ideal of \( R \) and \( \sigma(P \cap R) = P \cap R \).
2. If \( Q \) is a prime ideal of \( R \) such that \( \sigma(Q) = Q \), then \( S(Q) \) is a prime ideal of \( S(R) \) and \( S(Q) \cap R = Q \).

Proof. The proof follows on the same lines as in Lemma (10.6.4) of [9]. □

Theorem 1.10. Let \( R \) be a Noetherian ring. Let \( \sigma \) be an automorphism of \( R \) such that \( R \) is a \( \sigma(\ast) \)-ring. Then \( R \) is a weak \( \sigma \)-rigid ring. Conversely a 2-primal weak \( \sigma \)-rigid ring is a \( \sigma(\ast) \)-ring.

Proof. See Theorem (5) of [2]. □

Theorem 1.11. Let \( R \) be a right Noetherian \( \mathbb{Q} \)-algebra. Let \( \sigma \) be an automorphism and \( \delta \) be a \( \sigma \)-derivation of \( R \) such that \( \sigma(\delta(a)) = \delta(\sigma(a)) \) for all \( a \in R \). Then \( e^{\delta} \) is an automorphism of \( T = R[[t; \delta]] \), the skew power series ring.

Proof. The proof is on the same lines as in [11] and in the non-commutative case on the same lines as in [4]. Hence forth we denote \( R[[t; \delta]] \) by \( T \). Let \( \sigma \) be an automorphism of a ring \( R \), and \( I \) be an ideal of \( R \) such that \( \sigma(I) = I \). Then it is easy to see that \( TI \subseteq IT \) and \( IT \subseteq TI \). Hence \( TI = IT \) is an ideal of \( T \).
Lemma 1.12. Let $R$ be a right Noetherian $\mathbb{Q}$-algebra. Let $\sigma$ be an automorphism and $\delta$ be a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Let $I$ be an ideal of $R$ such that $\sigma(I) = I$. Then $I$ is $\delta$-invariant if and only if $IT$ is $\delta^{14}$-invariant.

Proof. See Lemma (2.5) of [3]. □

Proposition 1.13. Let $R$ be a ring and $T$ as usual. Then:

(1) $P \in \text{MinSpec}(T)$ implies that $P \cap R \in \text{MinSpec}(R)$ and $P = (P \cap R)T$.

(2) $U \in \text{MinSpec}(R)$ with $\sigma(U) = U$ implies that $UT \in \text{MinSpec}(T)$.

Proof. See Lemma (2.5) of [1]. □

2 Main Results

Proposition 2.1. Let $R$ be a ring. Then $R$ is an IFP-ring implies that $P(R)$ is completely semiprime.

Proof. Since $R$ is an IFP-ring. So, by Proposition (1.5) of [12] $R$ is 2-primal implies that $P(R)$ is completely semiprime. □

Proposition 2.2. Let $R$ be a right Noetherian IFP-ring which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ such that $R$ is a weak $\sigma$-rigid ring and $\delta$ a $\sigma$-derivation of $R$. Then $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in \text{MinSpec}(R)$.

Proof. Let $U \in \text{MinSpec}(R)$. Since $P(R)$ is completely semiprime by Proposition (2.1). So by Proposition (2.1) of [3] we have $\sigma(U) = U$. Now let $T = \{a \in U \mid \text{such that } \delta^k(a) \in U \text{ for all integers } k \geq 1\}$. Then $T$ is a $\delta$-invariant ideal of $R$. Hence it is easy to show that $\delta(U) \subseteq U$ by Proposition (2.1) of [3]. □

Lemma 2.3. Let $R$ be a right Noetherian IFP-ring which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ such that $R$ is a weak $\sigma$-rigid ring and $\delta$ a $\sigma$-derivation of $R$. Then

(1) If $U$ is a minimal prime ideal of $R$, then $O(U)$ is a minimal prime ideal of $O(R)$ and $O(U) \cap R = U$.

(2) If $P$ is a minimal prime ideal of $O(R)$, then $P \cap R$ is a minimal prime ideal of $R$.

Proof. Since every IFP-ring is 2-primal and a 2-primal weak $\sigma$-rigid ring is $\sigma(*)$-ring by Theorem (1.10). Rest is obvious by using Lemma (2.2) of [3]. □
Theorem 2.4. Let $R$ be a right Noetherian IFP-ring, which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ such that $R$ is a weak $\sigma$-rigid ring and $\delta$ be a $\sigma$-derivation of $R$. Then $P \in \text{Ass}(O(R)O(R))$ if and only if there exists $U \in \text{Ass}(R_R)$ such that $O(P \cap R) = P$ and $P \cap R = U$.

Proof. The proof follows on the same lines as in Theorem (A) of [3]. We give a sketch.

$R$ being right Noetherian implies that $\text{Ass}(R_R) = \mathcal{A}(R)$. Now $R$ is a weak $\sigma$-rigid IFP ring, therefore, Proposition (2.2) implies that $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in \text{MinSpec}(R)$. So $O(U)$ is an ideal of $O(R)$. Now $fU = 0$. Therefore $fO(R)U \subseteq fUO(R) = 0$, i.e. $U \subseteq P \cap R$. But it is clear that $P \cap R \subseteq U$. Thus $P \cap R = U$.

Conversely let $U = \text{Ann}(cR) = \text{Assas}(cR)$, $c \in R$ and $R$ is right Noetherian implies that $\text{Ass}(R_R) = \mathcal{A}(R)$. Now it can be easily seen that $O(U) = \text{Ann}(chO(R))$ for all $h \in O(R)$. Therefore $O(U) = \text{Ann}(cO(R)) = \text{Assas}(cO(R))$. □

References

