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A NOTE ON PRIME IDEALS OF IFP-RINGS AND THEIR EXTENSIONS

Smarti Gosani[1,*], **V. K. Bhat**[2]

[1] *Department of Applied Sciences and Humanities*
Model Institute of Engineering and Technology,
Jammu 181122, India
e-mail: smarti.gosani@gmail.com

[2] *School of Mathematics*
Shri Mata Vaishno Devi University
Jammu 182320, India
e-mail: Vijaykumarbhat2000@yahoo.com

Abstract

Let R be a ring, σ an automorphism of R and δ a σ-derivation of R. Let further σ be such that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$, where $N(R)$ is the set of nilpotent elements of R. We recall that a ring R is called an IFP-ring if for $a, b \in R$, $ab = 0$ implies $aRb = 0$. In this paper we study the associated prime ideals of Ore extension $R[x; \sigma, \delta]$ and we prove the following in this direction:

Let R be a right Noetherian IFP-ring, which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ is the field of rational numbers), σ and δ as above. Then P is an associated prime ideal of $R[x; \sigma, \delta]$ (viewed as a right module over itself) if and only if there exists an associated prime ideal U of R such that $(P \cap R)[x; \sigma, \delta] = P$ and $P \cap R = U$.

1 Introduction

Notation: We follow the notation and conventions of [3]. All rings are associative with 1. For any subset J of a right R-module M, annihilator of J is denoted by $Ann(J)$. $Spec(R)$ denotes the set of prime ideals of R, the set of associated prime ideals of R (viewed as a right module over itself) is denoted by $Ass(R_R)$. $MinSpec(R)$ denotes the set of minimal prime ideals of R. Let R

*Corresponding author

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be a right Noetherian ring. For any uniform right R-module J, the assassinator of J is denoted by $Assas(J)$. Let M be a right R-module. Consider the set $\{Assas(J) \mid J$ is a uniform right R-submodule of $M\}$. We denote this set by $\mathbb{A}(M_R)$.

Remark 1.1. If R is viewed as a right module over itself, we note that $Ass(R_R) = \mathbb{A}(R_R)$ ($5Y$ of Goodearl and Warfield [5]).

Ore Extensions: Let R be a ring and σ an endomorphism of R. Recall that a σ-derivation of R is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all a, $b \in R$. In case σ is the identity map, δ is called just a derivation of R.

The Ore extension (or the skew polynomial ring) over R in an indeterminate x is: $R[x;\sigma,\delta] = \{f(x) = x^n a_n + ... + a_0 \mid a_i \in R\}$ with $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. This definition of non-commutative polynomial rings was first introduced by Ore 1933, who combined earlier ideas of Hilbert (in the case $\delta = 0$) and Schlessinger (in the case $\sigma = 1$). We denote the Ore extension $R[x;\sigma,\delta]$ by $O(R)$. An ideal I of a ring R is called σ-invariant if $\sigma(I) = I$ and is called δ-invariant if $\delta(I) \subseteq I$. If an ideal I of R is σ-invariant and δ-invariant, then $I[x;\sigma,\delta]$ is an ideal of $O(R)$ and as usual we denote it by $O(I)$.

Definition 1.2. A ring R is called 2-primal if and only if $P(R) = N(R)$, where $P(R)$ is the prime radical of R and $N(R)$ is the set of nilpotent elements of R (a familiar property of commutative rings). Some of the fundamental properties of 2-primal rings are developed in [6], [12] and [13]. (N. B. The terminology is not uniform: 2-primal rings are called "N-rings" in [6], and, under an equivalent definition, called "weakly symmetric" in [13]).

An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$, where $a \in R$.

Weak σ-rigid rings and IFP-rings:

Definition 1.3. (Kwak [8]). Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 1.4. Let $R = \mathbb{Z}[\sqrt{-2}]$. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(a + b\sqrt{-2}) = a - b\sqrt{-2}$. Then R is a $\sigma(*)$-ring.

Ouyang in [10] introduced weak σ-rigid rings, where σ is an endomorphism of ring R. These rings are related to 2-primal rings.

Definition 1.5. (Ouyang [10]). Let R be a ring and σ an endomorphism of R such that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$. Then R is called a weak σ-rigid ring.

Example 1.6. Assume that $W_1[F]$ is the first Weyl algebra over a field F of characteristic zero. Then $W_1[F] = F[\mu, \lambda]$, the polynomial ring with indeterminates μ and λ with $\lambda\mu = \mu\lambda + 1$.
Now let R be the ring $\begin{pmatrix} W_1[F] & W_1[F] \\ 0 & W_1[F] \end{pmatrix}$. Consider the following element in R:$\begin{pmatrix} \mu & \lambda \\ 0 & 0 \end{pmatrix}$. Now the prime radical $P(R)$ of R is $\begin{pmatrix} 0 & W_1[F] \\ 0 & 0 \end{pmatrix}$. Define an endomorphism $\sigma : R \to R$ by $\sigma\Big(\begin{pmatrix} \mu & \lambda \\ 0 & 0 \end{pmatrix}\Big) = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}$. Then R is a weak σ-rigid ring.

Definition 1.7. (Shin [12]). Let R be a ring. Then R is called an IFP-ring (or Ring with Insert Factory Property) if for a, $b \in R$, $ab = 0$ implies $aRb = 0$. Also known as IFP-ring.

Example 1.8. (1) Let $R = \Big\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; a, b \in \mathbb{Z}\Big\}$.

The only matrices A and B satisfying $AB = 0$ are of the type $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$; $a, b \in \mathbb{Z}$.

i.e., $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$.

Now for all $K = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in R$,

$$AB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned}\text{implies } AKB &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\Big(\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\Big) \\ &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & db \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

This implies R is an IFP-ring.

(2) Let $R = \Big\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}; a, b \in \mathbb{Z}\Big\}$.

Then the only matrices A and B satisfying $AB = 0$ are of the type $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$; $a, b \in \mathbb{Z}$.

Now let a, b, c and $d \neq 0$ then for all $K = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in R$

$$AB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

But $AKB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$

$$= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \right)$$

$$= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & db \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & adb \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This implies R is not an IFP-ring.

(3) (Example (5.3) of [12]). Let $F = \mathbb{Z}_2(y)$ be the field of rational functions over $\mathbb{Z}_2$ with y an indeterminate. Consider the ring $R = \{f(x) \in F[x] \mid xy + yx = 1\}$. Then clearly R is a domain, so it is reduced and hence an IFP-ring.

Reduced rings (i.e., rings without nonzero nilpotent elements) are obviously IFP-rings, right (left) duo rings are IFP-rings by ([12], Lemma 1.2). Shin showed that IFP-rings are 2-primal in ([12], Theorem 1.5), and so reduced rings are 2-primal.

Lemma 1.9. *Let R be a ring. Let σ be an automorphism of R.*
1. If P is a prime ideal of $S(R)$ such that $x \notin P$, then $P \cap R$ is a prime ideal of R and $\sigma(P \cap R) = P \cap R$.
2. If Q is a prime ideal of R such that $\sigma(Q) = Q$, then $S(Q)$ is a prime ideal of $S(R)$ and $S(Q) \cap R = Q$.

Proof. The proof follows on the same lines as in Lemma (10.6.4) of [9]. □

Theorem 1.10. *Let R be a Noetherian ring. Let σ be an automorphism of R such that R is a $\sigma(*)$- ring. Then R is a weak σ-rigid ring. Conversely a 2-primal weak σ-rigid ring is a $\sigma(*)$-ring.*

Proof. See Theorem (5) of [2]. □

Theorem 1.11. *Let R be a right Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism and δ be a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $e^{t\delta}$ is an automorphism of $T = R[[t;\delta]]$, the skew power series ring.*

Proof. The proof is on the same lines as in [11] and in the non-commutative case on the same lines as in [4]. □ Hence forth we denote $R[[t;\delta]]$ by T. Let σ be an automorphism of a ring R, and I be an ideal of R such that $\sigma(I) = I$. Then it is easy to see that $TI \subseteq IT$ and $IT \subseteq TI$. Hence $TI = IT$ is an ideal of T.

Lemma 1.12. *Let R be a right Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism and δ be a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Let I be an ideal of R such that $\sigma(I) = I$. Then I is δ-invariant if and only if IT is $e^{t\delta}$-invariant.*

Proof. See Lemma (2.5) of [3]. □

Proposition 1.13. *Let R be a ring and T as usual. Then:*

(1) $P \in MinSpec(T)$ implies that $P \cap R \in MinSpec(R)$ and $P = (P \cap R)T$.

(2) $U \in MinSpec(R)$ with $\sigma(U) = U$ implies that $UT \in MinSpec(T)$.

Proof. See Lemma (2.5) of [1]. □

2 Main Results

Proposition 2.1. *Let R be a ring. Then R is an IFP-ring implies that $P(R)$ is completely semiprime.*

Proof. Since R is an IFP-ring. So, by Proposition (1.5) of [12] R is 2-primal implies that $P(R)$ is completely semiprime. □

Proposition 2.2. *Let R be a right Noetherian IFP-ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R. Then $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in MinSpec(R)$.*

Proof. Let $U \in MinSpec(R)$. Since $P(R)$ is completely semiprime by Proposition (2.1). So by Proposition (2.1) of [3] we have $\sigma(U) = U$. Now let $T = \{a \in U \mid \text{such that } \delta^k(a) \in U \text{ for all integers } k \geq 1\}$. Then T is a δ-invariant ideal of R. Hence it is easy to show that $\delta(U) \subseteq U$ by Proposition (2.1) of [3]. □

Lemma 2.3. *Let R be a right Noetherian IFP-ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R. Then*

(1) If U is a minimal prime ideal of R, then $O(U)$ is a minimal prime ideal of $O(R)$ and $O(U) \cap R = U$.

(2) If P is a minimal prime ideal of $O(R)$, then $P \cap R$ is a minimal prime ideal of R.

Proof. Since every IFP-ring is 2-primal and a 2-primal weak σ-rigid ring is $\sigma(*)$-ring by Theorem (1.10). Rest is obvious by using Lemma (2.2) of [3]. □

Theorem 2.4. *Let R be a right Noetherian IFP-ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ be a σ-derivation of R. Then $P \in Ass(O(R)_{O(R)})$ if and only if there exists $U \in Ass(R_R)$ such that $O(P \cap R) = P$ and $P \cap R = U$.*

Proof. The proof follows on the same lines as in Theorem (A) of [3]. We give a sketch.

R being right Noetherian implies that $Ass(R_R) = \mathbb{A}(R_R)$. Now R is a weak σ-rigid IFP ring, therefore, Proposition (2.2) implies that $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in MinSpec(R)$. So $O(U)$ is an ideal of $O(R)$. Now $fU = 0$. Therefore $fO(R)U \subseteq fUO(R) = 0$, i.e. $U \subseteq P \cap R$. But it is clear that $P \cap R \subseteq U$. Thus $P \cap R = U$.

Conversely let $U = Ann(cR) = Assas(cR)$, $c \in R$ and R is right Noetherian implies that $Ass(R_R) = \mathbb{A}(R_R)$. Now it can be easily seen that $O(U) = Ann(chO(R))$ for all $h \in O(R)$. Therefore $O(U) = Ann(cO(R)) = Assas(cO(R))$. □

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