Abstract

In this paper we consider a problem of signal processing where the signal is expressed by a fractional Ornstein-Uhlenbeck process in general form. An explicit form of the signal is derived from a fractional Langevin equation. A method of $L^2$-approximation is used to find the approximate estimate for the state of the fractional signal and the convergence to the optimal estimate is established.

1 Introduction

It is known that the Ornstein-Uhlenbeck plays a crucial role in telecommunication as an only stationary Gaussian Markov signal with white noise. But a Gaussian non-Markovian signal is also important in some context where the signal leaves a long time influence upon its behavior. A good candidate for expressing this signal property is a fractional Brownian noise. In this paper we consider a problem of signal processing where the signal is a fractional Ornstein-Uhlenbeck by introducing an approximation approach.

Key words: Fractional Ornstein-Uhlenbeck signal, $L^2$-approximation approach, fractional Brownian motion. 2010 MS: 60H, 93E05. 2010 AMS Mathematics classification: 60H, 93E05.

1
1.1 Fractional Brownian motion

A fractional Brownian motion of Mandelbrot form is a centered Gaussian process \((W^H_t, t \geq 0)\) with covariance function \(R(s, t)\) given by

\[
R(s, t) = E(W^H_s W^H_t) = \frac{1}{2} (s^{2H} + t^{2H} + |t-s|^{2H}),
\]

where \(H\) is a parameter called Hurst index, \(0 < H < 1\).

In the case where \(H = \frac{1}{2}\), \(W^H_t\) becomes a usual standard Brownian motion.

The process \(W^H_t\) can be decomposed as

\[
W^H_t = C_H (U_t + B^H_t),
\]

where \(U_t\) is a stochastic process with absolutely continuous trajectory and \(C_H\) is a constant depending only on \(H\), \(B^H_t = \int_0^t (t-s)^\alpha dW_s\) with \(\alpha = H - \frac{1}{2}\).

We know that \(W^H_t\) is a process of long memory with \(H \neq \frac{1}{2}\). In (1.2) this property focuses at the second term \(B^H_t\) and by this reason, \(B^H_t\) is called a fractional Brownian motion of Liouville form. In this paper we consider fractional noise associated with \(B^H_t\). The problem is how to get the optimal state estimation for a fractional signal that is a general fractional Ornstein-Uhlenbeck process \((X_t, t \geq 0)\) satisfying the following equation

\[
dX_t = (a(t)X_t + b(t))dt + \sigma dB^H_t,
\]

where \(H > 1/2\), from an observation \(Y_t\) given by

\[
dY_t = h(X_t)dt + dV_t,
\]

where \(V_t\) is a standard Brownian motion independent of \(B^H_t\), \(h_t = h(X_t)\) is a process such that

\[
E \int_0^t h_s^2 ds < \infty, \text{ for every } t \geq 0.
\]

1.2 Approximation approach

The fractional Brownian motion \(B^H_t\) is not a semimartingale, so a fractional signal driven by \(B^H_t\) as \(X_t\) in (1.3) cannot be solved by the traditional Ito calculus.

An \(L^2\)-approximation approach has been introduced in [2] where a process \(B^{H,\epsilon}_t\) is considered instead of \(B^H_t\):

\[
B^{H,\epsilon}_t = \int_0^t (t-s + \epsilon)^\alpha dW_s, \quad \alpha = H - \frac{1}{2}.
\]

A calculation says to us that \(B^{H,\epsilon}_t\) is in fact a semimartingale.

\[
dB^{H,\epsilon}_t = \alpha \varphi_t^\epsilon dt + \epsilon^\alpha dW_t,
\]
where
\[ \phi_t = \int_0^t (t - s + \varepsilon)\alpha^{-1}dW_s, \quad \alpha = H - \frac{1}{2}. \]

And as shown in [2] we have the following fundamental result on \(L^2\)-convergence of semimartingales \(B^H,\varepsilon_t\).

**Result:** \(B^H,\varepsilon_t\) converges to \(B_t\) in \(L^2(\Omega)\) when \(\varepsilon \to 0\) and we have
\[ \sup_{0 \leq t \leq T} \|B^H,\varepsilon_t - B_t\|_{L^2} \leq K(\alpha)e^{1/2+\alpha}, \tag{1.7} \]
where \(K(\alpha)\) is a constant depending only on \(\alpha = H - 1/2\).

Moreover a new approach to stochastic integration and stochastic differential equations driven by \(B^H_t\) is given in [3] (refer also to [3]-[9]).

## 2 General fractional Ornstein-Uhlenbeck signal

### 2.1 Approximate signal equation

Consider again the equation
\[ dX_t = (a(t)X_t + b(t))dt + \sigma dB^H_t, \quad H > 1/2, \tag{2.1} \]
where \(0 \leq t \leq T\), coefficients \(a(t)\) and \(b(t)\) are deterministic continuous function on \([0, T]\).

It is a generalization of fractional stochastic Langevin equation studied in [6] and [7], where our \(L^2\)-approximation method has been applied to find its solution. As shown in [10] the solution of (2.1) is a \(L^1\)-limit of that of an approximate equation. Now we prove that it is also a \(L^2\)-limit.

By replacing \(B^H_t\) by \(B^H,\varepsilon_t\) we obtain the approximate equation for the signal \(X_t\) as follows
\[ dX^\varepsilon_t = (a(t)X^\varepsilon_t + b(t) + \alpha\phi^\varepsilon_t)dt + \sigma\varepsilon dW_t, \tag{2.2} \]
where \(0 \leq t \leq T, \quad H > 1/2.\)

### 2.2 Approximate equation

Equation (2.2) can be rewritten as follows
\[ dX^\varepsilon_t = (a(t)X^\varepsilon_t + b(t) + \alpha\phi^\varepsilon_t)dt + \sigma\varepsilon dW_t. \tag{2.3} \]
A method of equation splitting introduced by us in [6, 7] can be applied to (2.3). We can write
\[ X^\varepsilon_t = X^\varepsilon_1(t) + X^\varepsilon_2(t), \quad 0 \leq t \leq T, \tag{2.4} \]
where
\[ dX_1^\epsilon(t) = a(t)X_1^\epsilon(t)dt + \sigma\epsilon^\alpha dW_t \]  \hspace{1cm} (2.5)

and
\[ dX_2^\epsilon(t) = (a(t)X_2^\epsilon(t) + b(t) + \alpha\varphi_t^\epsilon)dt. \]  \hspace{1cm} (2.6)

Equation (2.5) is a simple stochastic linear equation of Langevin type and its solution is
\[ X_1^\epsilon(t) = e^{\int_0^t a(u)du}(X_1^\epsilon(0) + \sigma\epsilon^\alpha \int_0^t e^{-\int_0^u a(u)du}dW_u). \]  \hspace{1cm} (2.7)

And the equation (2.6) is an ordinary differential equation for every fixed \( \omega \) and its solution is
\[ X_2^\epsilon(t) = e^{\int_0^t a(u)du}[X_0^\epsilon + \int_0^t b(s)e^{-\int_0^s a(u)du}ds + \sigma \int_0^t \varphi_s^\epsilon e^{-\int_0^s a(u)du}dW_s]. \]  \hspace{1cm} (2.8)

Now combining (2.4), (2.7) and (2.8) and noticing that \( \alpha\varphi_s^\epsilon ds + \epsilon^\alpha dW_s = dB_H^s, \epsilon \)
we can write the approximate signal \( X_t^\epsilon \) in the form
\[ X_t^\epsilon = X_1^\epsilon(t) + X_2^\epsilon(t) \]
\[ = e^{\int_0^t a(u)du}[X_0 + \int_0^t b(s)e^{-\int_0^s a(u)du}ds + \sigma \int_0^t e^{-\int_0^s a(u)du}dB_H^s, \epsilon], \]  \hspace{1cm} (2.9)

where \( X_0 \) is assumed a random variable such that \( E|X_0|^2 < \infty \).

3 **Convergence to the exact solution**

We can see that the equation (2.1) satisfies all conditions of Theorem of existence and uniqueness for solution of a fractional stochastic differential equation given in [3]. We will prove that the approximate signal \( X_t^\epsilon \) converges to the fractional \( X_t \) that is the exact solution of (2.1). Consider two equations
\[ dX_t = (a(t)X_t + b(t))dt + \sigma dB_t^H, \]
\[ dX_t^\epsilon = (a(t)X_t^\epsilon + b(t))dt + \sigma dB_t^{H,\epsilon}. \]

3.1 **Theorem 3.1**

\( X_t^\epsilon \) converges to \( X_t \) in \( L^2(\Omega) \) uniformly with respect to \( t \in [0, T] \).
Proof. We have

\[ X_t - X_t^\epsilon = a(t) \int_0^t (X_s - X_s^\epsilon) ds + \sigma (B_t^H - B_t^{H,\epsilon}). \]

Then

\[ \|X_t - X_t^\epsilon\| \leq M \left\| \int_0^t (X_s - X_s^\epsilon) ds \right\| + \sigma \|B_t - B_t^{H,\epsilon}\|, \quad (3.1) \]

where \(\| \cdot \|\) denote for \(L^2\)-norm and \(\|a(t)\| \leq M\) for \(t \in [0, T]\), \(M > 0\) due to the fact that \(a(t)\) is a continuous function.

In account of (1.7) we can see from (3.1) that

\[ \|X_t - X_t^\epsilon\| \leq M \int_0^t \|X_s - X_s^\epsilon\| ds + \sigma K(\alpha) \epsilon^{1+\alpha}, \quad 0 \leq t \leq T. \quad (3.2) \]

Applying the Gronwall’s lemma to (3.2) we get

\[ \|X_t - X_t^\epsilon\| \leq \sigma K(\alpha) \epsilon^{1+\alpha} e^{-at} \quad (3.3) \]

and then

\[ \sup_{0 \leq t \leq T} \|X_t - X_t^\epsilon\| \leq \sigma K(\alpha) \epsilon^{1+\alpha} e^{-aT} \text{ for } a > 0 \]

\[ \sup_{0 \leq t \leq T} \|X_t - X_t^\epsilon\| \leq \sigma K(\alpha) \epsilon^{1+\alpha} \text{ for } a < 0 \]

So \(X_t^\epsilon \to X_t\) in \(L^2(\Omega)\) uniformly with respect to \(t \in [0, T]\). \(\square\)

3.2 Corollary 3.1

It follows from Theorem 3.1 and the formula (2.9) that the exact signal \(X_t\) can be explicitly expressed as

\[ X_t = e^{\int_0^t a(u) du} (X_0 + \int_0^t \int_0^s b(v) e^{-\int_0^v a(u) du} dv ds + \sigma \int_0^t e^{-\int_0^s a(u) du} dB_s^H). \quad (3.4) \]

4 Best state estimate for signal \(X_t\)

4.1 Approximation for best state estimate

Consider now an approximate model for state estimate of the signal \(X_t^\epsilon\) form the observation \(Y_t\): Signal \(X_t^\epsilon\):

\[ dX_t^\epsilon = (a(t) X_t^\epsilon + b(t)) dt + \sigma dB_t^{H,\epsilon}. \quad (4.1) \]

Observation \(Y_t\):

\[ dY_t = h(X_t^\epsilon) dt + dV_t. \quad (4.2) \]
The model (4.1) - (4.2) can be rewritten as follows

\[ dX_t^\epsilon = \left( a(t)X_t^\epsilon + b(t) + \alpha \varphi_t^\epsilon \right) dt + \sigma \epsilon dW_t, \]  

(4.3)

\[ Y_t^\epsilon = \int_0^t h(X_s^\epsilon) ds + V_t. \]  

(4.4)

where \( W_t \) and \( V_t \) are two independent standard Brownian motion. Let \( \mathcal{F}_t^Y \) be the observation \( \sigma \)-algebra, that is the algebra generated by all random variables \( Y_s \) for \( s \leq t \):

\[ \mathcal{F}_t^Y = \sigma(Y_s, 0 \leq s \leq t). \]

Also, \( \mathcal{F}_t^{Y^*} \) is denoted for the approximate observation \( \sigma \)-algebra:

\[ \mathcal{F}_t^{Y^*} = \sigma(Y_s^\epsilon, 0 \leq s \leq t). \]

The best state estimation for approximate signal \( X_t^\epsilon \) denoted by \( \hat{X}_t^\epsilon \) based on observation information given by \( \mathcal{F}_t^Y \):

\[ \hat{X}_t^\epsilon = E(X_t|\mathcal{F}_t^Y). \]  

(4.5)

Denote by \( \nu_t \) the innovation process that is a \( \mathcal{F}_t^{Y^*} \)-martingale:

\[ \nu_t = Y_t^\epsilon - \int_0^t \hat{h}_s^\epsilon ds, \]  

(4.6)

where \( \hat{h}_s = h(X_s) = E(h(X_s)|\mathcal{F}_s^Y), 0 \leq s \leq t \) and by \( H_t^\epsilon \) the following expression

\[ H_t^\epsilon = a(t)X_t^\epsilon + b(t) + \alpha \varphi_t^\epsilon. \]  

(4.7)

Now we are in position to apply the FKK (Fujisaki-Kallianpur-Kunita) (see [11]) equation to \( \hat{X}_t^\epsilon \) and we have

**Theorem 4.1** The best state estimate \( \hat{X}_t^\epsilon \) is given by the following equation

\[ \hat{X}_t^\epsilon = \hat{X}_0^\epsilon + \int_0^t \hat{X}_s^\epsilon H_s^\epsilon ds + \int_0^t \left[ \hat{X}_s^\epsilon \hat{h}_s^\epsilon - \hat{X}_s^\epsilon \hat{h}_s^\epsilon \right] d\nu_s, \]  

(4.8)

where the notation \( \wedge \) stands for the best state estimate.

### 4.2 Best state estimation for the exact signal \( X_t \)

Now we have to find

\[ \hat{X}_t = E(X_t|\mathcal{F}_t^Y), \]  

(4.9)

where the signal \( X_t \) is given by (3.4).

Consider the best approximate state \( \hat{X}_t^\epsilon = E(X_t^\epsilon|\mathcal{F}_t^{Y^*}) \).

Put \( \epsilon = 1/n, n = 1, 2 \ldots \) and denote \( X^{(n)} \) for \( X_t^\epsilon \) with \( \epsilon = 1/n \).
Then \( \widehat{X}_t = X_t^{(n)} = E(X_t|\mathcal{F}_t^{(n)}) \) where \( \mathcal{F}_t^{(n)} = \mathcal{F}_t^{(n)} \) is the \( \sigma \)-algebra generated by \( (X_0, B_s^{(n)}, V_s, s \leq t) \) with

\[
B_t^{(n)} = B_{t}^{H,1/n} = \int_0^t (t - s - 1/n)^{\alpha}dW_s. \tag{4.10}
\]

By a change of variable we have

\[
B_t^{(n)} = \int_0^{t - 1/n} (t - u)^{\alpha}dW_u = B_{t-1/n} \quad \text{and} \quad B_s^{(n)} = B_{s-1/n}. \tag{4.11}
\]

Therefore \( \sigma \)-algebras \( \mathcal{F}_t^{(n)} = \sigma(X_0, B_{s-1/n}, V_s, s \leq t), n = 1, 2, \ldots \) form an increasing filtration and \( \mathcal{F}_t^{(n)} \uparrow \mathcal{F}_t^{Y} \).

By applying the elementary inequality

\[
|a + b|^2 \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2
\]

we can see

\[
E|E(X_t|\mathcal{F}_t^{(n)}) - E(X_t|\mathcal{F}_t^{Y})|^2 \leq \frac{1}{2}E|E(X_t^{(n)} - X_t|\mathcal{F}_t^{(n)})|^2 + \frac{1}{2}E|E(X_t|\mathcal{F}_t^{(n)}) - E(X_t|\mathcal{F}_t^{Y})|^2
\]

\[
\leq \frac{1}{2}E|X_t^{(n)} - X_t|^2 + \frac{1}{2}E|E(X_t|\mathcal{F}_t^{(n)}) - E(X_t|\mathcal{F}_t^{Y})|^2. \tag{4.12}
\]

In the last side of (4.12) we see that when \( n \to \infty \) the first term tends to 0 by Theorem 3.1 for \( \epsilon = 1/n \) and the second term converges to 0 as well because of a Levy theorem of convergence of conditional expectation. Finally we can state

**Theorem 4.2:** \( \widehat{X}_t \) can be considered as \( L^2 = \lim_{n \to \infty} X_t^{(n)} \) when \( n \to \infty \)

\[
\widehat{X}_t = L^2 = \lim_{n \to \infty} E(X_t^{(n)}|\mathcal{F}_t^{(n)}). \tag{4.13}
\]

**Acknowledgements** This research is funded by Vietnam National Foundation for Science and Technology Development NAFOSTED under Grant No 101.02-2011.12.

**References**


Fractional Ornstein-Uhlenbeck signal processing


