THE STABILITY OF SOLUTIONS OF GENERALIZED QUASI-EQUILIBRIUM PROBLEMS

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Abstract
In this paper, we study the generalized quasi-equilibrium problems and establish some results on the stability of solutions of generalized quasi-equilibrium problems and its applications.

1 Introduction
Throughout this paper $X, Z$ and $Y$ are supposed to be real topological locally convex Hausdorff spaces, $D \subset X, K \subset Z$ are nonempty subsets. Given multivalued mappings $P_1 : D \to 2^D, P_2 : D \to 2^D, Q : K \times D \to 2^K$ and $F : K \times D \times D \to 2^Y$, we are interested in the following problems:

Find $\bar{x} \in D$ such that

$\bar{x} \in P_1(\bar{x})$

and

$0 \in F(y, \bar{x}, t)$, for all $t \in P_2(\bar{x})$ and $y \in Q(\bar{x}, t)$.

This problem is called the generalized quasi-equilibrium problem of type II.

In the problems, the multivalued mappings $P_1, P_2$ and $Q$ are constraints mappings and $F$ is an utility multivalued mapping that are often determined by equalities and inequalities, or by inclusions, not inclusions and intersections of other multivalued mappings, or by some relations in product spaces. This

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70
problem involves many problems in optimization theory as special cases, such as optimal control problems, Minty variational inequalities, Nash equilibrium problems... The optimal control problem is following:

**Example 1.1.** Let $\Omega$ be an open bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and the boundary $\Gamma$ of class $C^1$. We consider the problem of finding a control function $u \in L^p(\Omega)$ with $1 < p < +\infty$ and a corresponding state $y \in W^{1,r}(\Omega)$ which 

$$
\text{minimize } J(y, u) = \int_{\Omega} L(x, y(x), u(x)) dx
$$

subject to 

$$
- \sum_{i,j=1}^{n} D_j ((a_{ij}(x)) . D_i y) + h(x, y) = u \text{ in } \Omega,
$$

$$
y = 0 \text{ on } \Gamma,
$$

with one of following contrains:

1). **Type 1: Mixed constraints**

$$
g_i (x, y(x), u(x)) \leq 0, \text{a.e. } x \in \Omega,
$$

$$
i = 1, ..., n;
$$

2). **Type 2: Homogeneous constraints**

$$
g(x, y(x)) \leq 0, \text{vi mi } x \in \Omega,

u(x) \in U, \text{ a.e., } x \in \Omega;
$$

3). **Type 3: Mixed and homogeneous constraints**

$$
g(x, y(x)) \leq 0, \text{ for all } x \in \Omega;

f_i (x, y(x), u(x)) \leq 0, \text{ a.e. } x \in \Omega,
$$

$$
i = 1, ..., n,
$$

where $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, h : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathodory function, $g_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is functions.

We defined the mappings $K(y, u) = Ay + h(., y) - u; G_i(y, u) = g_i(., y, u)$. When $g_i(., y, u) \in C(\Omega)$, we defined the mapping 

$$
\phi_i(y, u) = \max_{x \in \Omega} g_i(x, y(x), u(x)).
$$

The problem (1)-(3) becomes:

$$
\text{minimize } J(y, u),
$$

subject to 

$$
K(y, u) = 0, \text{ and } \phi_i(y, u) \leq 0,
$$

$$
i = 1, n.
The stability of solutions of generalized quasi-equilibrium problems

We set

\[ F(y, u, z, w) = J(y, u) - J(z, w) + \mathbb{R}_+; \]
\[ G(y, u, z, w) = \left( K(y, u), \prod_{i=1}^n \Phi_i(y, u) - \mathbb{R}_+ \right). \]

The above problems equivalent with the problems: Find \((\bar{y}, \bar{u}) \in W_0^{1,r}(\Omega) \times L^p(\Omega)\) such that

\[ 0 \in F(\bar{y}, \bar{u}, z, w) \times \left( K(y, u), \prod_{i=1}^n \Phi_i(y, u) - \mathbb{R}_+ \right), \]

which means that,

\[ J(\bar{y}, \bar{u}) \leq J(z, w) \]

for all \((z, w) \in W_0^{1,r}(\Omega) \times L^p(\Omega); \]
\[ K(\bar{y}, \bar{u}) = 0; \]
\[ \Phi_i(\bar{y}, \bar{u}) \leq 0, i = 1, \ldots, m. \]

This optimal are studied by Bui Trong Kien [3].

In [5], Nguyen Xuan Tan and Nguyen Thi Quynh Anh showed the sufficient conditions for the existence of solutions of generalized quasi-equilibrium problems, one of them is the following:

**Theorem 1.1.** The following conditions are sufficient for \((GEP)_{II}\) to have a solution:

i) \(D\) is a nonempty convex compact subset;

ii) \(P_1 : D \rightarrow 2^D\) is a multivalued mapping with a nonempty closed fixed point set \(D_0 = \{x \in D \mid x \in P_1(x)\}\) in \(D\);

iii) \(P_2 : D \rightarrow 2^D\) is a multivalued mapping with \(P_2^{-1}(x)\) open and the convex hull of \(P_2(x)\) of \(P_2(x)\) is contained in \(P_1(x)\) for each \(x \in D\);

iv) For any fixed \(t \in D\), the set

\[ B = \{x \in D \mid 0 \notin F(y, x, t), \text{ for some } y \in Q(x, t)\} \]

is open in \(D\);

v) \(F : K \times D \times D \rightarrow 2^Y\) is a \(Q - KKM\) multivalued mapping.

Our aim is to finding sufficient conditions for solutions mapping to be stable.
2 Preliminaries and Definitions

Throughout this paper, as in the introduction, by $X, Z, W$ and $Y$ we denote real topological locally convex Hausdorff spaces. Given a subset $D \subseteq X$, we consider a multivalued mapping $F : D \to 2^Y$. The domain and the graph of $F$ are denoted and defined by

$$\text{dom}F = \{x \in D | F(x) \neq \emptyset\},$$
$$\text{Gr}(F) = \{(x, y) \in D \times Y | y \in F(x)\},$$

respectively.

We recall that $F$ is said to be a closed mapping if the graph $\text{Gr}(F)$ of $F$ is a closed subset in the product space $X \times Y$ and it is said to be a compact mapping if the closure $\text{cl}F(D)$ of its range $F(D)$ is a compact set in $Y$.

A multivalued mapping $F : D \to 2^Y$ is said to be upper (lower) semicontinuous, briefly: u.s.c (respectively, l.s.c) at $\bar{x} \in D$ if for each open set $V$ containing $F(\bar{x})$ (respectively, $F(\bar{x}) \cap U \neq \emptyset$), there exists an open set $U$ of $\bar{x}$ that $F(x) \subseteq V$ (respectively, $F(x) \cap U \neq \emptyset$) for each $x \in U$ and $F$ is said to be u.s.c (l.s.c) on $D$ if it is u.s.c (respectively, l.s.c) at any point $x \in D$. These notions and definitions can be found in [2].

Let $D \subseteq X, K \subseteq Y$. The following proposition show the need and sufficient conditions to multivalued mappings be upper (lower) semicontinuous.

**Proposition 2.1.** ([4]) Assume $F : D \to 2^Y$ is a multivalued mappings with compact valueds. Then, $F$ is lower semicontinuous at $x \in D$ iff for all $y \in F(x)$ and for all net $\{x_\alpha\}$ in $D x_\alpha \to x$, exists a net $\{y_\alpha\}$ which satisfying $y_\alpha \in F(x_\alpha)$ for all $\alpha, y_\alpha \to y$.

**Proposition 2.2.** ([6]) If the multivalued mappings $F$ with $F^{-1}(x)$ open in $D$ then $F$ is lower semicontinuous.

**Proposition 2.3.** ([1]) If $F : D \to 2^K$ is an upper semicontinuous multivalued mapping with closed valueds, then it is closed. Ngc li, if $F$ is a closed mapping and $K$ is compact, then $F$ is an upper semicontinuous mapping.

3 The stability of solutions of generalized quasi-equilibrium problems

Let $X, Z, D, K, Y, C$ as the above sections, $\Lambda, \Gamma, \Sigma$ be real topological locally convex Hausdorff spaces, the mappings $P_i : D \times \Lambda \to 2^D$ with $i = 1, 2, Q : D \times D \times \Gamma \to 2^K$ and $F : K \times D \times D \times \Sigma \to 2^Y$. We are interested the generalized quasi-equilibrium problems dependent on parameter: Find $\bar{x} \in P_1(\bar{x}, \lambda)$ such that $0 \in F(y, \bar{x}, t, \mu)$ for all $t \in P_2(\bar{x}, \lambda), y \in Q(\bar{x}, t, \gamma)$. 
Theorem 3.1. Let \((\lambda_0, \gamma_0, \mu_0) \in \Lambda \times \Gamma \times \Sigma\). Assume that:

1) \(P_1\) is an upper semicontinuous multivalued mappings with compact values;
\(P_2\) is a lower semicontinuous multivalued mappings;

2) \(Q\) is an lower semicontinuous multivalued mappings with compact valueds;

3) The set \(A = \{(y, x, \lambda, \gamma, \mu) \mid x \in E(\lambda), 0 \in F(y, x, t, \gamma) \text{ for all } t \in P_2(x, \lambda), y \in Q(x, t, \mu)\}\) is closed.

Then,

i) \(M(\lambda_0, \gamma_0, \mu_0)\) is closed,

ii) \(M\) is upper semicontinuous multivalued mappings at \((\lambda_0, \gamma_0, \mu_0)\).

Proof Firstly, we proof that \(M\) is closed at \((\lambda_0, \gamma_0, \mu_0)\). We assume that \(M\) is not closed, it means existing the net \((x_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (x_0, \lambda_0, \gamma_0, \mu_0)\), with \(x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha), x_0 \notin M(\lambda_0, \gamma_0, \mu_0)\). From \(x_\alpha \in E(\lambda_\alpha)\) and the closedness of \(E\), it follows that \(x_0 \in E(\lambda_0)\). Since \(x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha)\), it follows that

\[0 \in F(y_\alpha, x_\alpha, t_\alpha, \mu_\alpha)\]

for all \(t_\alpha \in P_2(x_\alpha, \lambda_\alpha), y_\alpha \in Q(x_\alpha, t_\alpha)\).

Moreover, \((y_\alpha, x_\alpha, t_\alpha) \in D\) is compact, without loss of generality, we may assume that \(y_\alpha \to y_0, x_\alpha \to x_0, t_\alpha \to t_0\). We have \((y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0)\), it implies \((y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0) \in A\). Since that, we have \(x_0 \in M(\lambda_0, \gamma_0, \mu_0)\). We have a contradiction. So, \(M\) is closed at \((\lambda_0, \gamma_0, \mu_0)\).

Next, we proof that the mappings \(M : \Lambda \times \Gamma \times \Sigma \to 2^D\) is upper semicontinuous at \((\lambda_0, \gamma_0, \mu_0)\). We assume that \(M\) is not upper semicontinuous at \((\lambda_0, \gamma_0, \mu_0)\). Then, exists the open set \(U\) contains \(M(\lambda_0, \gamma_0, \mu_0)\) such that for any net \(\{(\lambda_\alpha, \gamma_\alpha, \mu_\alpha)\}\) converges to \((\lambda_0, \gamma_0, \mu_0)\), exists \(x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha), x_\alpha \notin U\). Since \(P_1\) is a multivalued mappings with compact valueds, it implies \(P_1\) (so, \(E\)) is a closed mappings. We may assume that \(x_\alpha \to x_0\), so \(x_0 \in E(\lambda_0)\). If \(x_0 \notin M(\lambda_0, \gamma_0, \mu_0)\) then there exists \(t_0 \in P_2(x_0, \lambda_0), y_0 \in Q(x_0, t_0, \gamma_0)\) such that

\[0 \notin F(y_0, x_0, t_0, \mu_0)\].

(6)

From \((x_\alpha, \lambda_\alpha) \to (x_0, \lambda_0)\), \(P_2\) is lower semicontinuous at \((x_0, \lambda_0)\), it implies that \(t_\alpha \in P_2(x_\alpha, \lambda_\alpha), t_\alpha \to t_0\). Moreover, \(Q\) is lower semicontinuous at \((x_0, t_0, \gamma_0)\),
so \( y_\alpha \in Q(x_\alpha, t_\alpha, \gamma_\alpha) \), \( y_\alpha \to y_0 \). Since \( x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha) \), we have \( 0 \in F(y_\alpha, x_\alpha, t_\alpha, \mu_\alpha) \). On the other hand, \( (y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0) \), \( x_\alpha \in E(\lambda_\alpha) \), \( t_\alpha \in P_2(x_\alpha, \lambda_\alpha) \), \( y_\alpha \in Q(x_\alpha, t_\alpha, \mu_\alpha) \), \( 0 \in F(y_\alpha, x_\alpha, t_\alpha, \mu_\alpha) \). In view of the closedness of \( A \), we obtain

\[
(y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0) \in A.
\]

This conflict with (6), we have \( M \) is an upper semicontinuous multivalued mapping. □

**Theorem 3.2.** Assume that:

1) \( E \) is a lower semicontinuous multivalued mappings at \( \lambda_0 \);

2) \( Q \) is an upper semicontinuous mapping with compact values;

3) \( P_2 \) is a closed multivalued mapping;

4) The set \( A = \{(y, x, t, \lambda, \gamma, \mu) \in D \times D \times D \times \Lambda \times \Gamma \times \Sigma \mid x \in P_1(x, \lambda), 0 \notin F(y, x, t, \lambda, \gamma, \mu), t \in P_2(x, \lambda), y \in Q(x, t, \mu)\} \) is closed.

Then, \( M \) is a lower semicontinuous multivalued mapping at \( (\lambda_0, \gamma_0, \mu_0) \).

**Proof** We assume that \( M \) is not lower semicontinuous at \( (\lambda_0, \gamma_0, \mu_0) \), i.e. there exists the net \( (\lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (\lambda_0, \gamma_0, \mu_0) \), \( x_0 \in M(\lambda_0, \gamma_0, \mu_0) \) such that \( x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha) \), \( x_\alpha \not\to x_0 \). By the lower semicontinuity of \( E \), \( x_0 \in E(\lambda_0) \), \( \lambda_\alpha \to \lambda_0 \), there exists \( x'_\alpha \in E(\lambda_\alpha) \), \( x'_\alpha \to x_0 \), \( x'_\alpha \notin M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha) \). Therefore, there exists \( t_\alpha \in P_2(x_\alpha, \lambda_\alpha) \), \( y_\alpha \in Q(x_\alpha, t_\alpha, \mu_\alpha) \) such that \( 0 \notin F(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \). One the other hand, \( Q \) is an upper semicontinuous mapping with compact valued, \( P_2 \) is a closed mapping, \( \{t_\alpha\} \subseteq D \), \( \{y_\alpha\} \subseteq K \) is 1 comparative compact. Hence, we may assume that \( y_\alpha \to y_0, t_\alpha \to t_0 \) and \( y_0 \in Q(x_0, t_0, \mu_0) \), \( t_0 \in P_2(x_0, \lambda_0) \). We have

\[
(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \in A, (y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0).
\]

So \( (y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0) \in A \), i.e.

\[
0 \notin F(y_0, x_0, t_0, \mu_0), x_0 \in P_1(x_0, \lambda_0), t_0 \in P_2(x_0, \lambda_0), y_0 \in Q(x_0, t_0, \mu_0).
\]

This is a contradiction with \( x_0 \in M(\lambda_0, \gamma_0, \mu_0) \). The proof are complete. □
Example 3.1. Consider in optimal control problem on times (see in Introduction):

Let $\Omega$ be open bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and the boundary $\Gamma$ of class $C^1$. We consider the problem of finding a control function $(u, \gamma, \mu) \in L^p(\Omega) \times \Gamma \times \Sigma$ with $1 < p < +\infty$ and a corresponding state $(y, \gamma, \mu) \in W^{1,r}(\Omega)$ which

$$\min J(y, u, \mu) = \int_{\Omega} L(x, y(x), u(x), \mu) \, dx$$

subject to

$$- \sum_{i,j=1}^n D_j (a_{ij}(x) \cdot D_i y) + h(x, y, \gamma) = u \, \text{in} \, \Omega,$$

$$y = 0 \, \text{on} \, \Gamma,$$

with one of following contrains:

1). Type 1: Mixed constraints

$$g_i (x, y(x), u(x), \gamma) \leq 0, \, \text{a.e.} \, x \in \Omega, \quad i = 1, \ldots, n;$$

2). Type 2: Homogeneous constraints

$$g(x, y(x), \gamma) \leq 0, \, \forall \, x \in \Omega,$$

$$u(x) \in U, \, \text{a.e.}, \, x \in \Omega;$$

3). Type 3: Mixed and homogeneous constraints

$$g(x, y(x), \gamma) \leq 0, \, \text{for all} \, x \in \Omega;$$

$$f_i (x, y(x), u(x), \gamma) \leq 0, \, \text{a.e.} \, x \in \Omega, \quad i = 1, \ldots, n;$$

Assume that

$$\frac{1}{n} > \frac{1}{r} \geq \frac{1}{p} - \frac{1}{n}. \quad (12)$$

$(u, \gamma) \in W^{1,r}(\Omega) \times \Gamma, (y, \gamma) \in W^{1,r}_0(\Omega) \times \Gamma$ is a solution of (8) iff

$$\int_{\Omega} \left( \sum_{i,j=1}^n a_{ij} D_i D_j \varphi \right) \, dx + \int_{\Omega} h(x, y, \gamma) \varphi \, dx = \langle u, \varphi \rangle \, \forall \, \varphi \in W^{1,r}_0(\Omega).$$

From (12) and Sobolev and Rellich Theorem, we have $L^p(\Omega) \hookrightarrow W^{1,r}(\Omega)$. Hence, $(u, \gamma) \in L^p(\Omega) \times \Gamma$, the equation (8) c duy nht nghim $(y, \gamma), y \in W^{1,r}_0(\Omega) \hookrightarrow C(\Omega)$. 
We define the mappings:

\[ K(y, u, \gamma) = Ay + h(., y, \gamma) - u, \]

\[ G_i(y, u, \gamma) = g_i(., y(x), u(x), \gamma) - \gamma \]

If \( g_i(., y, u, \gamma) \in C(\bar{\Omega}) \), the mapping

\[ \phi_i(y, u, \gamma) = \max_{x \in \Omega} g_i(x, y(x), u(x), \gamma) \]

is well defined. The above problem becomes:

\[
\begin{align*}
\text{minimize} & \quad J(y, u, \mu), \\
\text{s.t.} & \quad K(y, u, \gamma) = 0, \quad \nu \phi(y, u, \gamma) \leq 0.
\end{align*}
\]

Setting

\[ F(y, u, z, w, \mu) = J(y, u, \mu) - J(z, w, \mu) + \mathbb{R}_+, \]

\[ G(y, u, z, w, \gamma) = (K(y, u, \gamma), \prod_{i=1}^n \Phi_i(y, u, \gamma) - \mathbb{R}_+) \]

The above problem equivalent with following problem: Find \( (\bar{y}, \bar{u}, \bar{\gamma}, \bar{\mu}) \in W_0^{1,r}(\Omega) \times \mathbb{R}^p(\Omega) \times \Gamma \times \Sigma \) such that

\[
0 \in F(\bar{y}, \bar{u}, z, w, \bar{\mu}) \times \left( K(\bar{y}, \bar{u}, \bar{\gamma}), \prod_{i=1}^n \Phi_i(\bar{y}, \bar{u}, \bar{\gamma}) - \mathbb{R}_+ \right),
\]

that is

\[
J(\bar{y}, \bar{u}, \bar{\mu}) \leq J(z, w, \bar{\mu})
\]

for all \( (z, w) \in W_0^{1,r}(\Omega) \times \mathbb{R}^p(\Omega) \);

\[
K(\bar{y}, \bar{u}, \bar{\gamma}) = 0, \quad \Phi_i(\bar{y}, \bar{u}, \bar{\gamma}) \leq 0, \quad i = 1, 2, ..., m.
\]

Assume that \( P_1, Q \) are as in Section 2.5. We call \( M : \Lambda \times \Gamma \times \Sigma \rightarrow W_0^{1,r}(\Omega) \times \mathbb{R}^p(\Omega) \) a solutions mapping of the optimal control problem: Find \( (\bar{y}, \bar{u}, \bar{\gamma}, \bar{\mu}) \in W_0^{1,r}(\Omega) \times \mathbb{R}^p(\Omega) \times \Gamma \times \Sigma \) such that

\[
J(\bar{y}, \bar{u}, \bar{\mu}) \leq J(z, w, \bar{\mu})
\]

for all \( z, w \in W_0^{1,r}(\Omega) \times \mathbb{R}^p(\Omega) \),

\[
K(\bar{y}, \bar{u}, \bar{\gamma}) = 0,
\]

\[
\Phi_i(\bar{y}, \bar{u}, \bar{\gamma}) \leq 0, \quad i = 1, 2, ..., m,
\]

for all \( w \in P_1(\bar{x}, \bar{\lambda}), z \in Q(\bar{x}, w) \).

To show that \( M \) be an upper (lower) semicontinuous multivalued mapping at \( (\bar{\gamma}, \bar{\mu}) \), we find conditions to ensure that \( A \) (in 3.1, 3.2) is closed. For example, if \( J \) is a continuous function and \( P_1, Q \) as in 3.1, then \( A \) is closed.
The stability of solutions of generalized quasi-equilibrium problems

References